DISSERTATIONS

ON

SUBJECTS OF SCIENCE

CONNECTED WITH

NATURAL THEOLOGY:

BEING THE CONCLUDING VOLUMES OF

THE NEW EDITION

OF

PALEY'S WORK.

BY

HENRY LORD BROUGHAM, F.R.S.,

AND MEMBER OF THE NATIONAL INSTITUTE OF FRANCE.

IN TWO VOLUMES.

VOL. II.

LONDON:

C. KNIGHT AND CO., 22, LUDGATE-STREET.

1839.

LONDON:
Printed by W. Clowes and Sons,
Stamford-street.

DISSERTATION

ON

THE ORIGIN OF EVIL.

The question which has more than any other harassed metaphysical reasoners, but especially theologians, and upon which it is probable that no very satisfactory conclusion will ever be reached by the human faculties in our present state, is the Origin and Sufferance of Evil. Its existence being always assumed, philosophers have formed various theories for explaining it, but they have also drawn very different inferences from it. The ancient Epicureans argued against the existence of the Deity, because they held that the existence of Evil either proved him to be limited in power or of a malignant nature; either of which imperfections is inconsistent with the first notions of a divine being. In this kind of reasoning they have been followed both by the atheists and the sceptics of later times. With both sects this is a favourite topic, the one drawing from it a conclusion against the existence of anything like what religious men call a superintending providence, the other using the topic as a fruitful source of doubts, and as helping them to involve the whole question in those clouds which are the proper element of their speculations. It is to be observed, however, that the sceptics make more use of the argument from evil than the dogmatical atheists; for as long as design is proved to exist in the universe, the malignity of the overruling principle, how painful soever to our contemplation, would, though fully admitted, offer no proof against that derived from the positive evidences of its existence. To the sceptic the consideration of evil has been supposed more favourable, although without much foundation; as it never can throw any doubt upon the grand fundamental truth of natural religion, although it certainly may unsettle men's minds as to some of the other doctrines, and create some hesitation as to believing in the attributes, when it has failed to surround the existence of a Deity with any obscurity. That the great practical sceptic of all, Bayle—he who carried into every branch of inquiry the power on which he most valued himself, of involving all subjects in doubt*—regarded the subject of evil as one of the great arsenals from whence his weapons were to be most chiefly drawn is undeniable. None of the articles in his famous Dictionary are more laboured than those in which he treats of this subject. *Manichean*, and still more *Paulician*, almost assume the appearance of formal treatises upon the question; and both *Marcionite* and *Zoroaster* treat of the same subject. All these articles are of considerable value; they contain the greater part of the learning upon the question; and they are distinguished by the acuteness of reasoning which was the other characteristic of their celebrated author.

Those ancient philosophers who did not agree with Epicurus in arguing from the existence of evil against the existence of a providence that superintended and influenced the destinies of the world, were put to no little difficulty in accounting for the fact which they did not deny, and yet maintaining the power of a divine ruler. The doctrine of a double principle, or of two divine beings of opposite

* The epithet of "cloud-compelling," νεφεληγερέτα Ζεύς, was the one he chiefly valued.

natures, one beneficent, the other mischievous, was the solution which one class of reasoners deemed satisfactory, and to which they held themselves driven by the phenomena of the universe. Others, unable to deny the existence of things which men denominate evil, both physical and moral, explained them in a different way. They maintained that physical evil only obtains the name from our own imperfect and vicious or feeble dispositions; that to a wise man there is no such thing; that we may rise superior to all such grovelling notions as make us dread or repine at any events which can befall the body; that pain, sickness, loss of fortune or of reputation, exile, death itself, are only accounted ills by a weak and pampered mind; that if we find the world tiresome, or woeful, or displeasing, we may at any moment quit it; and that therefore we have no right whatever to call any suffering connected with existence on earth an evil, because almost all sufferings can be borne by a patient and firm mind; since if the situation we are placed in becomes either intolerable, or upon the whole more painful than agreeable, it is our own fault that we remain in it. But these philosophers further took a view of the question which especially applied to moral evil. They considered that nothing could be more groundless than to suppose that if there were no evil there could be any good in the world; and they illustrated this position by asking how we could know anything of temperance, fortitude, or justice, unless there were such things as excess, cowardice, and injustice.

These were the doctrines of the Stoics, from whose sublime and impracticable philosophy they seemed naturally enough to flow. Aulus Gellius relates that the last-mentioned argument was expounded by Chrysippus, in his work upon Providence.* The answer given by Plutarch seems quite sufficient: "As well might you say that Achilles could not have a fine head of hair unless Thersites had been bald; or that one man's limbs could not be all sound if another had not the gout." In truth, the Stoical doctrine proceeds upon the assumption that all virtue is only the negative of vice; and is as absurd, if indeed it be not the very same absurdity, as the doctrine which should deny the existence of affirmative or positive truths, resolving them all into the opposites of negative propositions. Indeed, if we even were to admit

* Aul. Gel. lib. vi. cap. 1.

this as an abstract position, the actual existence of evil would still be unnecessary to the idea, and still more to the existence, of good. For the conception of evil, the bare idea of its possibility, would be quite sufficient, and there would be no occasion for a single example of it.

The other doctrine, that of two opposite principles, was embraced by most of the other sects, as it should seem, at some period or other of their inquiries. Plato himself, in his later works, was clearly a supporter of the system; for he held that there were at least two principles, a good and an evil; to which he added a third, the moderator or mediator between them. Whether this doctrine was, like many others, imported into Greece from the East, or was the natural growth of the schools, we cannot ascertain. Certain it is that the Greeks themselves believed it to have been taught by Zoroaster in Asia, at the least five centuries before the Trojan war; so that it had an existence there long before the name of philosophy was known in the western world. Zoroaster's doctrine agreed in every respect with Plato's; for beside Oromazes, the good, and Arimanius,* the evil principle, he taught that

* Called in the East Ormusd, and Ahriman.

there was a third, or mediatory one, called Mithras. That it never became any part of the popular belief in Greece or Italy is quite clear. All the polytheism of those countries recognised each of the gods as authors alike of good and evil. Nor did even the chief of the divinities, under whose power the rest were placed, offer any exception to the general rule; for Jupiter not only gave good from one urn and ill from another, but he was also, according to the barbarous mythology of classical antiquity, himself a model at once of human perfections, and of human vices.

After the purer light of the Christian religion had made some way towards supplanting the ancient polytheism, the doctrine of two principles was broached; first by Marcion, who lived in the time of Adrian and Antoninus Pius, early in the second century; and next by Manes, a hundred years later. He was a Persian slave, who was brought into Greece, where he taught this doctrine, since known by his name, having learnt it, as is said, from Scythianus, an Arabian. The Manichean doctrines, afterwards called also Paulician, from a great teacher of them in the seventh century, were, like almost all the heresies in the primitive church, soon mixed up with gross impurities of sacred rites as well as extravagant absurdities of creed. The Manicheans were, probably as much on this account as from the spirit of religious intolerance, early the objects of severe persecution; and the Code of Justinian itself denounces capital punishment against any of the sect, if found within the Roman dominions.

It must be confessed that the theory of two principles, when kept free from the absurdities and impurities which were introduced into the Manichean doctrine, is not unnaturally adopted by men who have no aid from the light of revelation, and who are confounded by the appearance of a world where evil and good are mixed together, or seem to struggle with one another, sometimes the one prevailing, and sometimes the other; and accordingly in all countries, in the most barbarous nations, as well as among the most refined, we find plain traces of reflecting men having been driven to this solution of the difficulty. It seems upon a superficial view to be very easily deducible from the phenomena; and as the idea of infinite power, with which it is manifestly inconsistent, does by no means so naturally present itself to the mind, as long as only a very great degree of power, a power which in comparison of all human force may be termed infinite, is the attribute with which the Deity is believed to be endued, the Manichean hypothesis is by no means so easily refuted. That the power of the Deity was supposed to have limits even in the systems of the most enlightened heathens is unquestionable. They, generally speaking, believed in the eternity of matter, and conceived some of its qualities to be so essentially necessary to its existence that no divine agency could alter them. They ascribed to the Deity a plastic power, a power not of creating or annihilating, but only of moulding, disposing, and moving matter. So over mind they generally gave him the like power, considering it as a kind of emanation from his own greater mind or essence, and destined to be reunited with him hereafter. Nay, over all the gods, and of superior potency to any, they conceived Fate to preside; an overruling and paramount necessity, of which they formed some dark conceptions, and to which the chief of all the gods was supposed to submit. It is, indeed, extremely difficult to state precisely what the philosophic theory of theology was in Greece and Rome, because the wide difference between the esoteric and exoteric doctrines, between the belief of the learned few and the popular superstition, makes it very difficult to avoid confounding the two, and lending to the former some of the grosser errors with which the latter abounded. Nevertheless we may rely upon what has been just stated, as conveying, generally speaking, the opinion of philosophers, although some sects certainly had a still more scanty measure of belief. But we shall presently find that in the speculations of the much more enlightened moderns, Christians of course, errors of a like kind are to be traced. They constantly argue the great question of evil upon a latent assumption, that the power of the Deity is restricted by some powers or qualities inherent in matter; notions analogous to that of fate are occasionally perceptible; not stated or expanded indeed into propositions, but influencing the course of the reasoning; while the belief of infinite attributes is never kept steadily in view, except when it is called in as requisite to refute the Manichean doctrine. Some observers of the controversy have indeed not scrupled to affirm that those of whom we speak are really Manicheans without knowing it;

and build their systems upon assumptions secretly borrowed from the disciples of Zoroaster, without ever stating those assumptions openly in the form of postulates or definitions.

The refutation of the Manichean hypothesis is extremely easy if we be permitted to assume that both the principles which it supposes are either of infinite power or of equal power. If they are of infinite power the supposition of their co-existence involves a contradiction in terms; for the one being in opposition to the other, the power of each must be something taken from that of the other; consequently neither can be of infinite power. If, again, we only suppose both to be of equal power, and always acting against each other, there could be nothing whatever done, neither good nor evil; the universe would be at a stand still; or rather no act of creation could ever have been performed, and no existence could be conceived beyond that of the two antagonist principles. Archbishop Tillotson's argument, properly speaking, amounts to this last proposition, and is applicable to equal and opposite principles, although he applies it to two beings, both infinitely powerful and counteracting one another. When he says that they would tie up each other's hands, he might apply this argument to such antagonist principles if only equal, although not infinitely powerful. The hypothesis of their being both infinitely powerful needs no such refutation; it is a contradiction in terms.* But it must be recollected that the advocates of the Manichean doctrine endeavour to guard themselves against this attack by contending, that the conflict between the two principles ends in a kind of compromise, so that neither has it all his own way; there is a mixture of evil admitted by the good principle, because else the whole would be at a stand still; while there is much good admitted by the evil principle, else nothing, either good or evil, would be done. Another answer is therefore required to this theory than what Tillotson, and his followers, have given.

First, we must observe that this reasoning of the Manicheans is "after the manner of men." It proceeds upon the analogy of what we see in mortal contentions; where neither party having the power to defeat the other, each is content to yield a little to his adversary, and so, by mutual concession, both are successful to some extent, and both to some extent disappointed. But in a speculation con-

* See Tillotson's Sermons, vol. ii. p. 690, fol.

cerning the nature of the Deity, there seems no place for such notions.

Secondly, the equality of power is not an arbitrary assumption ; it seems to follow from the existence of the two opposing principles. For if they are independent of one another as to existence, which they must needs be else one would immediately destroy the other, so must they also, in each particular instance, be independent of each other, and also equal each to the other, else one would have the mastery, and the influence of the other could not be perceived. To say that in some things the good principle prevails and in others the evil, is really saying nothing more than that good exists here and evil there. It does not further the argument one step, nor give anything like an explanation. For it must always be borne in mind that the whole question respecting the Origin of Evil proceeds upon the assumption of a wise, benevolent, and powerful Being having created the world. The difficulty, and the only difficulty, is, how to reconcile existing evil with such a Being's attributes ; and if the Manichean only explains this by saying the good Being did what is good, and another and evil Being did what is bad in the universe, he really tells us nothing more than the fact; he does not apply his explanation to the difficulty; and he supposes the existence of a second Deity gratuitously and to no kind of purpose.

But, *thirdly*, in whatever light we view the hypothesis, it seems exposed to a similar objection, namely, of explaining nothing in its application, while it is wholly gratuitous in itself. It assumes, of course, that creation was the act of the good Being; and it also assumes that Being's goodness to have been perfect, though his power is limited. Then as he must have known the existence of the evil principle and foreseen the certainty of misery being occasioned by his existence, why did he voluntarily create sentient beings to put them, in some respects at least, under the evil one's power, and thus be exposed to suffering? The good Being, according to this theory, is the remote cause of the evil which is endured, because but for his act of creation the evil Being could have had no subjects whereon to work mischief; so that the hypothesis wholly fails in removing, by more than one step, the difficulty which it was invented to solve.

Fourthly, there is no advantage gained to the argument by supposing two Beings, rather than one

Being of a mixed nature. The facts lead to this supposition just as naturally as to the hypothesis of two principles. The existence of the evil Being is as much a detraction from the power of the good one, as if we only at once suppose the latter to be of limited power, and that he prefers making and supporting creatures who suffer much less than they enjoy, to making no creatures at all. The supposition that he made them as happy as he could, and that not being able to make them less miserable, he yet perceived that upon the whole their existence would occasion more happiness than if they never had any being at all, will just account for the phenomena as well as the Manichean theory, and will as little as that theory assume any malevolence in the power which created and preserved the universe. If, however, it be objected that this hypothesis leaves unexplained the fetters upon the good Being's power, the answer is obvious; it leaves those fetters not at all less explained than the Manichean theory does; for that theory gives no explanation of the existence of a counteracting principle, and it assumes both an antagonist power to limit the Deity's power, and a malevolent principle to set the antagonist power in motion; whereas our supposition assumes no malevolence at all, but only a restraint upon the divine power.

Fifthly, this leads us to another and most formidable objection. To conceive the eternal existence of one Being infinite in power, self-created and creating all others, is by no means impossible. Indeed, as everything must have had a cause, nothing we see being by possibility self-created, we naturally mount from particulars to generals, until finally we rise to the idea of a first cause, uncreated, and self-existing, and eternal. If the phenomena compel us to affix limits to his goodness, we find it impossible to conceive limits to the power of a creative, eternal, self-existing principle. But even supposing we could form the conception of such a Being having his power limited as well as his goodness, still we can conceive no second Being independent of him. This would necessarily lead to the supposition of some third Being, above and antecedent to both, and the creator of both—the real first cause—and then the whole question would be to solve over again, *ποθεν το κακον*;—Why these two antagonist Beings were suffered to exist by the great Being of all?

The Manichean doctrine, then, is exposed to every objection to which a theory can be obnoxious. It is gratuitous; it is inapplicable to the facts; it supposes more causes than are necessary; it fails to explain the phenomena, leaving the difficulties exactly where it found them. Nevertheless such is the theory, how easily soever refuted when openly avowed and explicitly stated, which in various disguises appears to pervade the explanations given of the facts by most of the other systems; nay, to form, secretly and unacknowledged, their principal groundwork. For it really makes very little difference in the matter whether we are to account for evil by holding that the Deity has created as much happiness as was consistent with "the nature of things," and has taken every means of avoiding all evil except "where it necessarily existed;" or at once give those limiting influences a separate and independent existence, and call them by a name of their own, which is the Manichean hypothesis.

The most remarkable argument on this subject, and the most distinguished both for its clear and well ordered statement, and for the systematic shape which it assumes, is that of Archbishop King. It is the great text-book of those who study this subject; and, like the famous legal work of Littleton it has found an expounder yet abler and more learned than the author himself. Bishop Law's commentary is full of information, of reasoning, and of explication; nor can we easily find anything valuable upon the subject which is not contained in the volumes of that work. It will, however, only require a slight examination of the doctrines maintained by these learned and pious men, to satisfy us that they all along either assume the thing to be proved, or proceed upon suppositions quite inconsistent with the infinite power of the Deity—the only position which raises a question, and which makes the difficulty that requires to be solved.

According to all the systems as well as this one, evil is of two kinds—physical and moral. To the former class belong all the sufferings to which sentient beings are exposed from the qualities and affections of matter independent of their own acts; the latter class consists of the sufferings of whatever kind which arise from their own conduct. This division of the subject, however, is liable to one serious objection; it comprehends under the second head a class of evils which ought more properly to be ranged under the first. Nor is this a mere question of classification: it affects the whole scope of the argument. The second of the above-mentioned classes comprehends both the physical evils which human agency causes, but which it would have no power to cause unless the qualities of matter were such as to produce pain, privation, and death; and also the moral evil of guilt which may possibly exist independent of material agency, but which, whether dependent or not upon that physical action, is quite separable from it, residing wholly in the mind. Thus a person who destroys the life of another produces physical evil by means of the constitution of matter, and moral evil is the source of his wicked action. The true arrangement then is this:—Physical evil is that which depends on the constitution of matter, or only is so far connected with the constitution of mind as that the nature and existence of a sentient being must be assumed in order to its mischief being felt. And this physical evil is of two kinds; that which originates in human action, and that which is independent of human action, befalling us from the unalterable course of nature. Of the former class are the pains, privations, and destruction inflicted by men upon one another; of the latter class are diseases, old age, and death. Moral evil consists in the crimes, whether of commission or omission, which men are guilty of—including under the latter head those sufferings which we endure from ill-regulated minds through want of fortitude or self-control. It is clear that as far as the question of the origin of evil is concerned, the first of these two classes, physical evil, depends upon the properties of matter, and the last upon those of mind. The second as well as the first subdivision of the physical class depends upon matter; because however ill-disposed the agent's mind may be, he could inflict the mischief only in consequence of the constitution of matter. Therefore, the Being who created matter enabled him to perpetrate the evil, even admitting that this Being did not by creating the mind also give rise to the evil disposition; and admitting that, as far as regards this disposition, it has the same origin with the evil of the second class, or moral evil, the acts of a rational agent.

It is quite true that many reasoners refuse to allow any distinction between the evil produced by natural causes and the evil caused by rational agents, whether as regards their own guilt, or the mischief it causes to others. Those reasoners deny that the creation of man's will and the endowing it with liberty explains anything; they hold that the creation of a mind whose will is to do evil, amounts to the same thing, and belongs to the same class, with the creation of matter whose nature is to give pain and misery. But this position, which involves the doctrine of Necessity, must, at the very least, admit of one modification. Where no human agency whatever is interposed, and the calamity comes without any one being to blame for it, the mischief seems a step, and a large step, nearer the creative or the superintending cause, because it is, as far as men go, altogether inevitable. The main tendency of the argument therefore is confined to physical evil; and this has always been found the most difficult to account for, that is to reconcile with the government of a perfectly good and powerful Being. It would indeed be very easily explained and the reconcilement would be readily made, if we were at liberty to suppose matter independent in its existence, and in certain qualities, of the divine control; but this would be to suppose the Deity's power limited and imperfect, which is just one horn of the Epicurean dilemma, "Aut vult et non potest;" and in assuming this, we do not so much beg the question as wholly give it up and admit we cannot solve the difficulty. Yet obvious as this is, we shall presently see that the reasoners who have undertaken the solution, and especially King and Law, under such phrases as "the nature of things,' and "the laws of the material universe," have been constantly, through the whole argument, guilty of this *petitio principii*, or rather this abandonment of the whole question, and never more so than at the very moment when they complacently plumed themselves upon having overcome the difficulty.

Having premised these observations for the purpose of clearing the ground and avoiding confusion in the argument, we may now consider what Archbishop King's theory is in both its parts; for there are in truth two distinct explanations, the one resembling an argument *à priori*, the other an argument *à posteriori*. It is, however, not a little remarkable that Bishop Law, in the admirable abstract or analysis which he gives of the archbishop's treatise at the end of his preface, begins with the second branch, omitting all mention of the first as if he considered it to be merely introductory matter; and yet his fourteenth note (t. cap. 1. s. 3.) shows that he was aware of its being an argument wholly independent of the rest of the reasoning; for he there says that the author had given one demonstration *à priori*, and that no difficulties raised by an examination of the phenomena, no objection *à posteriori*, ought to overrule it, unless these difficulties are equally certain and clear with the demonstration, and admit of no solution consistent with that demonstration.

The necessity of a first cause being shown, and it being evident that therefore this cause is uncreated and self-existent, and independent of any other, the conclusion is next drawn that its power must be infinite. This is shown by the consideration that there is no other antecedent cause, and no other principle which was not created by the first cause, and consequently which was not of inferior power; therefore there is nothing which can limit the power of the first cause; and there being no limiter or restrainer, there can be no limitation or restriction.

Again, the infinity of the Deity's power is attempted to be proved in another way. The number of possible things is infinite; but every possibility implies a power to do the possible thing; and as one possible thing implies a power to do it, an infinite number of possible things implies an infinite power. Or as Descartes and his followers put it, we can have no idea of anything that has not either an actual or a possible existence; but we have an idea of a Being of infinite perfection; therefore he must actually exist; for otherwise there would be one perfection wanting, and so he would not be infinite, which he either is actually or possibly. It is needless to remark that this whole argument, whatever may be said of the former one, is a pure fallacy, and a *petitio principii* throughout. The Cartesian form of it is the most glaringly fallacious, and indeed exposes itself; for by that reasoning we might prove the existence of a fiery dragon or any other phantom of the brain. But even King's more concealed sophism is equally absurd. What ground is there for saying that the number of possible things is infinite? he adds, "at least in power," which means either nothing or only that we have the power of conceiving an infinite number of possibilities. But because we can conceive or fancy an infinity of possibilities, does it follow that there actually exists this infinity? The whole argument is unworthy of a moment's consideration. The other is more plausible, that restriction implies a restraining power. But even this is not satisfactory when closely examined. For although the first cause must be self-existent and of eternal duration, we only are driven by the necessity of supposing a cause whereon all the argument rests, to suppose one capable of causing all that actually exists; and therefore to extend this inference and suppose that the cause is of infinite power seems gratuitous. Nor is it necessary to suppose another power limiting its efficacy, if we do not find it necessary to suppose its own constitution and essence such as we term infinitely powerful. However, after noticing this manifest defect in the fundamental part of the argument, that which infers infinite power, let us for the present assume the position to be proved, either by these or by any other reasons, and see if the structure raised upon it, is such as can stand the test of examination.

Thus, then, an infinitely powerful being exists, and he was the creator of the universe; but to incline him towards the creation there could be no possible motive of happiness to himself, and he must, says King, have either sought his own happiness or that of the universe which he made. Therefore his own design must have been the communication of happiness to the creature. He could only desire to exercise his attributes without, or externally to himself, which before creating other beings he could not do. But this could only gratify his nature, which wants nothing, being perfect in itself, by communicating his goodness and providing for the happiness of other sentient beings created by him for this purpose. Therefore, says King, "it manifestly follows that the world is as well as it could be made by infinite power and goodness; for since the exercise of the divine power and the communication of his goodness are the ends for which the world is formed, there is no doubt but God has attained these ends." And again, "If then any thing inconvenient or incommodious be now, or was from the beginning in it, that certainly could not be hindered or removed even by infinite power, wisdom, and goodness."

Now certainly no one can deny, that if God be infinitely powerful and also infinitely good, it must follow that whatever looks like evil, either is not really evil, or that it is such as infinite power could not avoid. This is implied in the very terms of the hypothesis. It may also be admitted that if the Deity's only object in his dispensation be the happiness of his creatures, the same conclusion follows even without assuming his nature to be infinitely good; for we admit what, for the purpose of the argument, is the same thing, namely, that there entered no evil into his design in creating or maintaining the universe. But all this really assumes the very thing to be proved. King gets over the difficulty and reaches his conclusion by saying, "The Deity could have only one of two objects—his own happiness or that of his creatures." The sceptic makes answer, "He might have another object, namely, the misery of his creatures;"—and then the whole question is, whether or not he had this other object; or, which is the same thing, whether or not his nature is perfectly good. It must never be forgotten, that unless evil exists, there is nothing to dispute about; *cadit quæstio.* The whole difficulty arises from the admission that evil exists; or that what we call evil exists. From this we inquire whether or not the author of it can be perfectly benevolent? or if he be, with what view he has created it? This assumes him to be infinitely powerful, or at least powerful enough to have prevented the evil; but indeed we are now arguing with the archbishop on the supposition that he has proved the Deity to be of infinite power. The sceptic rests upon his dilemma, and either alternative, limited power or limited goodness, satisfies him.

It is quite plain, therefore, that King has assumed the thing to be proved in his first argument, or argument *à priori.* For he proceeds upon the postulates that the Deity is infinitely good, and that he only had human happiness in view when he made the world. Either supposition would have served his purpose; and making either would have been taking for granted the whole matter in dispute. But he has assumed both; and it must be added, he has made his assumption of both as if he was only laying down a single position. This part of the work is certainly more slovenly than the rest. It is the third section of the first chapter.

It is certainly not from any reluctance to admit the existence of evil that the learned author and his able commentator have been led into this inconclusive course of reasoning. We shall nowhere find more striking expositions of the state of things in this respect, nor more gloomy descriptions of our condition, than in their celebrated work. "Whence so many inaccuracies (says the archbishop) in the work of a most good and most powerful God?

Whence that perpetual war between the very elements, between animals, between men? Whence errors, miseries, and vices, the constant companions of human life from its infancy? Whence good to evil men, evil to the good? If we behold anything irregular in the works of men, if any machine serve not the end it were made for, if we find something in it repugnant to itself or others, we attribute that to the ignorance, impatience, or malice of the workman. But since these qualities have no place in God, how come they to have place in anything? Or why does God suffer his works to be deformed by them?"—(Chap. ii. s. 3.) Bishop Law in his admirable preface still more cogently puts the case: "When I inquire how I got into this world, and came to be what I am, I am told that an absolutely perfect being produced me out of nothing, and placed me here on purpose to communicate some part of his happiness to me, and to make me in some measure like himself. This end is not obtained—the direct contrary appears—find myself surrounded with nothing but perplexity, want, and misery—by whose fault I know not—how to better myself I cannot tell. What notions of good and goodness can this afford me? What ideas of religion? What hopes of a future state? For if God's aim in producing me be entirely unknown, if it be neither his glory (as some will have it), which my present state is far from advancing, nor mine own good, which the same is equally inconsistent with, how know I what I am to do here, or indeed in what manner I must endeavour to please him? Or why should I endeavour it at all? For if I must be miserable in this world, what security have I that I shall not be so in another too, (if there be one,) since, if it were the will of my Almighty Creator, I might (for aught I see) have been happy in both."—(Pref. viii.)

The question thus is stated. The difficulty is raised in its full and formidable magnitude by both these learned and able men; that they have signally failed to lay it by the argument *à priori* is plain. Indeed it seems wholly impossible ever to answer by an argument *à priori* any objection whatever which arises altogether out of the facts made known to us by experience alone, and which are therefore in the nature of contingent truths, resting upon contingent evidence, while all demonstrations *à priori* must necessarily proceed upon mathematical truths. Let us now see if their labours have been more successful in applying to the solution of the difficulty the reasoning *à posteriori.*

Archbishop King divides evil into three kinds—imperfection, natural evil, and moral evil—including under the last head all the physical evils that arise from human actions, as well as the evil which consists in the guilt of those actions.

The existence of imperfection is stated to be necessary, because everything which is created and not self-existent must be imperfect; consequently every work of the Deity, in other words, everything but the Deity himself, must have imperfection in its nature. Nor is the existence of some beings which are imperfect any interference with the attributes of others. Nor the existence of beings with many imperfections any interference with others having preeminence. The goodness of the Deity therefore is no impugned by the existence of various orders of created beings more or less approaching to perfection. His creating none at all would have left the universe less admirable and containing less happiness than it now does. Therefore, the act of mere benevolence which called those various orders into existence is not impeached in respect of goodness any more than of power by the variety of the attributes possessed by the different beings created.

Upon this argument it may be observed that much of it is solid; but one part is ill-considered. The goodness of the Deity is well shown not to be impugned by the imperfection of any creatures; but the necessity of the imperfection is not proved by merely saying that all created beings, all which are not self-existent, must be imperfect. They might by possibility be perfect in every respect except their being created. This argument commits the great paralogism of substituting all existing imperfections for the one imperfection of not having self-existence. The main stress of the question, however, has never rested so much upon mere evils of imperfection, unless we reckon dissolution and death among them, which King by no means does.

He now proceeds to grapple with the real difficulty of the question. And it is truly astonishing to find this acute metaphysician begin with an assumption which entirely begs that question. As imperfection, says he, arises from created beings having been made out of nothing, so natural evils arise "from all natural things having a relation to matter, and on this account being necessarily subject to natural evil." As long as matter is subject to motion, it must be the subject of generation and corruption. "These and all other natural evils," says the author, "are so necessarily connected with the material origin of things that they cannot be separated from it, and thus the structure of the world either ought not to have been formed at all, or these evils must have been tolerated without any imputation on the divine power and goodness." Again, he says, "corruption could not be avoided without violence done to the laws of motion and the nature of matter." Again, "All manner of inconveniences could not be avoided because of the imperfection of matter and the nature of motion. That state of things were therefore preferable which was attained with the fewest and the least inconveniences." Then follows a kind of menace, "And who but a very rash indiscreet person will affirm that God has not actually made choice of this?"—when every one must perceive that the bare propounding of the question concerning evil calls upon us to exercise this temerity and commit this indiscretion.—(Chap. iv. s. 1, div. 7.) He then goes into more detail as to particular cases of natural evil; but all are handled in the same way. Thus death is explained by saying that the bodies of animals are a kind of vessels which contain fluids in motion, and being broken, the fluids are spilt and the motions cease; "because by the native imperfection of matter it is capable of dissolution, and the spilling and stagnation must necessarily follow, and with it animal life must cease."—(Chap. iv. s. 3.) Disease is dealt with in like manner. "It could not be avoided unless animals had been made of a quite different frame and constitution."—(Chap. iv. s. 7.) The whole reasoning is summed up in the concluding section of this part, where the author somewhat triumphantly says, "The difficult question then, whence comes evil? is not unanswerable. For it arises from the very nature and constitution of created beings, and could not be avoided without a contradiction."—(Chap. iv. s. 9.) To this the commentary of Bishop Law adds (Note 41), "that natural evil has been shown to be, in every case, unavoidable, without introducing into the system a greater evil."

It is certain that many persons, led away by the authority of a great name, have been accustomed to regard this work as a text-book, and have appealed to Archbishop King and his learned commentator as having solved the question. So many men have referred to the "Principia" as showing the motions of the heavenly bodies, who never read, or indeed could read, a page of that immortal work. But no man ever did open it who could read it and find himself disappointed in any one particular; the whole demonstration is perfect; not a link is wanting; nothing is assumed. How different the case here! We open the work of the prelate and find it from first to last a chain of gratuitous assumptions, and, of the main point, nothing whatever is either proved or explained. Evil arises, he says, from the nature of matter. Who doubts it? But is not the whole question why matter was created with such properties as of necessity to produce evil? It was impossible, says he, to avoid it consistently with the laws of motion and matter. Unquestionably; but the whole dispute is upon those laws. If indeed the laws of nature, the existing constitution of the material world, were assumed as necessary, and as binding upon the Deity, how is it possible that any question ever could have been raised? The Deity having the power to make those laws, to endow matter with that constitution, and having also the power to make different laws and to give matter another constitution, the whole question is, how his choosing to create the present existing order of things—the laws and the constitution which we find to prevail—can be reconciled with perfect goodness. The whole argument of the archbishop assumes that matter and its laws are independent of the Deity; and the only conclusion to which the inquiry leads us is that the Creator has made a world with as little of evil in it as the nature of things—that is, as the laws of nature and matter—allowed him; which is nonsense, if those laws were made by him, and leaves the question where it was, or rather solves it by giving up the omnipotence of the Creator, if these laws were binding upon him.

It must be added, however, that Dr. King and Dr. Law are not singular in pursuing this most inconclusive course of reasoning. Thus Dr. J. Clarke, in his treatise on natural evil, quoted by Bishop Law (Note 32), shows how mischiefs arise from the laws of matter; and says this could not be avoided "without altering those primary laws, that is, making it something else than what it is, or changing it into another form; the result of which would only be to render it liable to evils of another kind against which the same objections would equally lie." So Dr. J. Burnett, in his discourses on evil, at the Boyle Lecture (vol. ii. p. 201), conceives that he explains death by saying, that the materials of which the body is composed "cannot last beyond seventy years, or thereabouts, and it was originally intended that we should die at that age." Pain, too, he imagines, is accounted for by observing that we are endowed with feelings, and that if we could not feel pain so neither could we pleasure (p. 202). Again he says that there are certain qualities which "in the nature of things matter is uncapable of" (p. 207). And as if he really felt the pressure of this difficulty, he at length comes to this conclusion, that life is a free gift, which we had no right to exact, and which the Deity lay under no necessity to grant, therefore we must take it with the conditions annexed (p. 210); which is undeniably true, but is excluding the discussion and not answering the question proposed. Nor must it be forgotten that some reasoners deal strangely with the facts. Thus Derham, in his "Physico-Theology," explaining the use of poison in snakes, first desires us to bear in mind that many venomous ones are of use medicinally in stubborn diseases, which is not true, and if it were would prove nothing, unless the venom, not the flesh, were proved to be medicinal; and then says, they are "scourges upon ungrateful and sinful men;" adding the truly astounding absurdity, "that the nations which know not God are the most annoyed with noxious reptiles and other pernicious creatures" (Book ix. c. 1); which if it were true would raise a double difficulty, by showing that one people was scourged because another had neglected to preach the gospel among them. Dr. J. Burnett, too, accounts for animals being suffered to be killed as food for man, by affirming that they thereby gain all the care which man is thus led to bestow upon them, and so are, on the whole, the better for being eaten. (Boyle Lecture, II. 207). But the most singular error has perhaps been fallen into by Dr. Sherlock, and the most unhappy—which yet Bishop Law has cited as a sufficient answer to the objection respecting death: "It is a great instrument of government, and makes men afraid of committing such villanies as the laws of their country have made capital" (Note 34). So that the greatest error in the criminal legislation of all countries forms part of the divine providence, and man has at length discovered, by the light of reason, the folly and the wickedness of using an instrument expressly created by divine Omniscience to be abused!

The remaining portion of King's work, filling the second volume of Bishop Law's edition, is devoted to the explanation of Moral Evil; and here although there are some sound and irrefragable doctrines explained, yet the gratuitous assumption of the "nature of things," and the "laws of nature," more or less pervade the whole as in the former parts of the Inquiry.

The fundamental position of the whole is, that man having been endowed with free will, his happiness consists in making due elections, or in the right exercise of that free will. Five causes are then given of undue elections, in which of course his misery consists as far as that depends on himself; these causes are, error, negligence, over-indulgence of free choice, obstinacy or bad habit, and the importunity of natural appetites; which last it must in passing be remarked belongs to the head of physical evil, and cannot be assumed in this discussion without begging the question. The great difficulty is then stated and grappled with, namely, how to reconcile these undue elections with divine goodness. The objector states that free will might exist without the power of making undue elections, by being suffered to range, as it were, only among lawful objects of choice. But the answer to this seems sound, that such a will would only be free in name; it would be free to choose among certain things, but would not be free-will. The objector again urges, that either the choice is free and may fall upon evil objects, against the goodness of God, or it is so restrained as only to fall on good objects, against freedom of the will. King's solution is, that more evil would result from preventing these undue elections than from suffering them, and so the Deity has only done the best he could in the circumstances; a solution obviously liable to the same objection as that respecting Natural Evil. There are three ways, says the archbishop, in which undue elections might have been prevented; not creating a free-agent—constant interference with his free-will—removing him to another state where he would not be tempted to go astray in his choice. A fourth mode may, however, be suggested,—creating a free-agent without any inclination to evil, or any temptation from external objects. When our author disposes of the second method, by stating that it assumes a constant miracle, as great in the moral as altering the course of the planets hourly would be in the material universe, nothing can be more sound or more satisfactory. But when he argues that our whole happiness consists in a consciousness of freedom of election, and that we should never know happiness were we restrained in any particular, it seems wholly inconceivable how he should have omitted to consider the prodigious comfort of a state in which we should be guaranteed against any error or impropriety of choice; a state in which we should both be unable to go astray and always feel conscious of that security. He, however, begs the question most manifestly in dealing with the two other methods stated, by which undue elections might have been precluded. "You would have freedom," says he, "without any inclination to sin; but it may justly be doubted if this is possible *in the present state of things*," (ch. v. s. 5, sub. 2); and again, in answering the question why God did not remove us into another state where no temptation could seduce us, he says: "It is plain that in *the present state of things* it is impossible for men to live without natural evils or the danger of sinning." (Ib.) Now the whole question arises upon the constitution of the present state of things. If that is allowed to be inevitable, or is taken as a datum in the discussion, there ceases to be any question at all.

The doctrine of a chain of being is enlarged upon, and with much felicity of illustration. But it only wraps up the difficulty in other words, without solving it. For then the question becomes this—Why did the Deity create such a chain as could not be filled up without misery? It is, indeed, merely restating the fact of evil existing; for whether we say there is suffering among sentient beings,—or the universe consists of beings more or less happy, more or less miserable,—or there exists a chain of beings varying in perfection and in felicity,—it is manifestly all one proposition. The remark of Bayle upon this view of the subject is really not at all unsound, and is eminently ingenious: "Would you defend a king who should confine all his subjects of a certain age in dungeons, upon the ground that if he did not, many of the cells he had built must remain empty?" The answer of Bishop Law to this remark is by no means satisfactory. He says it assumes that more misery than happiness exists. Now, in this view of the question, the balance is quite immaterial. The existence of any evil at all raises the question as much as the preponderance of evil over good, because the question conceives a perfectly good Being, and asks how such a Being can have permitted any evil at all. Upon this part of the subject both King and Law have fallen into an error which recent discoveries place in a singularly clear light. They say that the argument they are dealing with would lead to leaving the earth to the brutes without human inhabitants. But the recent discoveries in Fossil Osteology have proved that the earth, for ages before the last 5,000 or 6,000 years, was left to the lower animals; nay, that in a still earlier period of its existence no animal life at all was maintained upon its surface. So that, in fact, the foundation is removed of the *reductio ad absurdum* attempted by the learned prelates.

A singular argument is used towards the latter end of the Inquiry. When the Deity, it is said, resolved to create other beings, He must of necessity tolerate imperfect natures in his handiwork, just as he must the equality of a circle's radii when he drew a circle. Who does not perceive the difference? The meaning of the word circle is that the radii are all equal; this equality is a necessary truth. But it is not shown that men could not exist without the imperfections they labour under. Yet this is the argument suggested by these authors while complaining (ch. v. s. 5, sub. 7, div. 7) that Lactantius had not sufficiently answered the Epicurean dilemma; it is the substitute propounded to supply that father's deficiency.—"When, therefore," says the archbishop, "matter, motion, and free will are constituted, the Deity must necessarily permit corruption of things and the abuse of liberty, or something worse, for these cannot be separated without a contradiction, and God is no more impotent than because he cannot separate equality of radii from a circle." (Ch. v. s. 5. subs. 7.) If he could not have created evil, he would not have been omnipotent; if he would not, he must let his power lie idle; and rejecting evil have rejected all the good. "Thus (exclaims the author with triumph and self-complacency) then vanishes this Herculean argument, which induced the Epicureans to discard the good Deity, and the Manicheans to substitute an evil one." (Ib. subs. 7. *sub fine*). Nor is the explanation rendered more satisfactory, or indeed more intelligible, by the concluding passage of all, in which we are told that "from a conflict of two properties, namely, omnipotence and goodness, evils necessarily arise. These attributes amicably conspire together, and yet restrain and limit each other." It might have been expected from hence that no evil at all should be found to exist. "There is a kind of struggle and opposition between them, whereof the evils in nature bear the shadow and resemblance. Here then, and no where else, may we find the primary and most certain rise and origin of evils." Such is this celebrated work; and it may safely be affirmed that a more complete failure to overcome a great and admitted difficulty,—a more unsatisfactory solution of an important question,—is not to be found in the whole history of metaphysical science.

Among the authors who have treated of this subject a high place is justly given to Archdeacon Balguy, whose work on Divine Benevolence is always referred to by Dr. Paley with great commendation. But certain it is that this learned and pious writer either had never formed to himself a very precise notion of the real question under discussion, namely, the compatibility of the appearances which we see and which we consider as evil, with a Being infinitely powerful as well as good; or he had in his mind some opinions respecting the divine nature, opinions of a limitary kind, which he does not state distinctly, although he constantly suffers them to influence his reasonings. Hence, whenever he comes close to the real difficulty he appears to beg the question. A very few instances of what really pervades the whole work, will suffice to show how unsatisfactory its general scope is, although it contains, like the treatise of Dr. King and Dr. Law's Commentary, many valuable observations on the details of the subject.

And first we may perceive that what he terms a "*Previous Remark,*" and desires the reader "to carry along through the *whole** proof of divine benevolence," really contains a statement that the difficulty is to be evaded and not met. "An intention of producing good (says he) will be sufficiently apparent in any particular instance if the thing considered can neither be changed nor taken away without loss or harm, *all other things continuing the same.*† Should you suppose *various*† things in the system changed *at once,*† you can neither judge of the possibility

* These italics are in the original, and no doubt they point our attention to the refutation at once.

† Sic in orig.

nor the consequences of the changes, having no degree of experience to direct you." Now assuredly this postulate makes the whole question as easy a one as ever metaphysician or naturalist had to solve. For it is no longer—Why did a powerful and benevolent Being create a world in which there is evil,—but only,—The world being given, how far are its different arrangements consistent with one another? According to this, the earthquake at Lisbon, Voltaire's favourite instance, destroyed thousands of persons, because it is in the nature of things that subterraneous vapours should explode, and that when houses fall on human beings they should be killed. Then if Dr. Balguy goes to his other argument, on which he often dwells, that if this nature were altered, we cannot possibly tell whether worse might not ensue; this, too, is assuming a limited power in the Deity, contrary to the hypothesis. It may most justly be said, that if there be any one supposition necessarily excluded from the whole argument, it is the fundamental supposition of the "Previous Remark," namely, "all other things continuing the same."

But see how this assumption pervades and paralyzes the whole argument, rendering it utterly inconclusive. The author is to answer an objection derived from the constitution of our appetites for food, and his reply is, that "we cannot tell how far it was *possible** for the stomachs and palates of animals to be differently formed, unless by some remedy worse than the disease." † Again, upon the question of pain: "How do we know that it was *possible** for the uneasy sensation to be confined to particular cases."‡ So we meet the same fallacy under another form, as evil being the result of "general principles." But no one has ever pushed this so far as Dr. Balguy, for he says, "that in a government so conducted many events are likely to happen contrary to the intention of its author."§ He now calls in the aid of chance, or accident—"It is probable," he says, "that God should be good, for evil is more likely to be *accidental** than appears from experience in the conduct of men."‖ Indeed his fundamental position of the Deity's benevolence is rested upon this foundation, that "*pleasures only** were intended, and that the pains are accidental consequences although the means of producing pleasures."¶ The same

* Sic in orig. † P. 34. ‡ P. 38. § P. 29.
‖ P. 21. ¶ P. 19.

recourse to accident is repeatedly had. Thus, "the events to which we are exposed in this imperfect state appear to be the *accidental** not natural effects of our frame and condition."† Now can any one thing be more manifest than that the very first notion of a wise and powerful Being excludes all such assumptions as things happening contrary to His intention; and that when we use the word chance or accident, which only means our human ignorance of causes, we at once give up the whole question, as if we said, "It is a subject about which we know nothing." So again as to power. "A good design is more *difficult** to be executed, and therefore more likely to be executed *imperfectly,** than an evil one, *i. e.*, with a mixture of effects foreign to the design and opposite to it."‡ This at once assumes the Deity to be powerless. But a general statement is afterwards made more distinctly to the same effect. "Most sure it is that he can do all things possible. But are we in any degree competent judges of the bounds of possibility."§ So again under another form nature is introduced as something different from its author, and offering limits to his power. "It is plainly not the method of

* Sic orig. † P. 113. ‡ P. 23. § P. 109.

nature to obtain her ends instantaneously."* Passing over such propositions as that "*useless*† evil is a thing never seen" (when the whole question is why the same ends were not attained without evil), and a variety of other subordinate assumptions contrary to the hypothesis, we may rest with this general statement, which almost every page of Dr. Balguy's book bears out, that the question which he has set himself to solve is anything rather than the real one touching the Origin of Evil; and that this attempt at a solution is as ineffectual as any of those which we have been considering.

Is then the question wholly incapable of solution, which all these learned and ingenious men have so entirely failed in solving? Must the difficulty remain for ever unsurmounted, and only be approached to discover that it is insuperable? Must the subject, of all others the most interesting for us to know well, be to us always as a sealed book, of which we can never know anything? From the nature of the thing—from the question relating to the operation of a power which, to our limited faculties, must ever be incomprehensible—there seems too much reason for believing that nothing precise or

* P. 112. † Sic orig.

satisfactory ever will be attained by human reason regarding this great argument; and that the bounds which limit our view will only be passed when we have quitted the encumbrances of our mortal state, and are permitted to survey those regions beyond the sphere of our present circumscribed existence. The other branch of Natural Theology, that which investigates the evidences of Intelligence and Design, and leads us to a clear apprehension of the Deity's power and wisdom, is as satisfactorily cultivated as any other department of science, rests upon the same species of proof, and affords results as precise as they are sublime. This branch will never be distinctly known, and will always so disappoint the inquirer as to render the lights of Revelation peculiarly acceptable, although even those lights leave much of it still involved in darkness—still mysterious and obscure.

Yet let us endeavour to suggest some possible explication, while we admit that nothing certain, nothing entirely satisfactory can be reached. The failure of the great writers whose works we have been contemplating may well teach us humility, make us distrust ourselves, and moderate within us any sanguine hopes of success. But they should not make us wholly despair of at least showing in what direction the solution of the difficulty is to be sought, and whereabouts it will probably be found situated, when our feeble reason shall be strengthened and expanded. For one cause of their discomfiture certainly has been their aiming too high, attempting a complete solution of a problem which only admitted of approximation, and discussion of limits.*

It is admitted on all hands that the demonstration is complete which shows the existence of intelligence and design in the universe. The structure of the eye and ear in exact conformity to the laws of optics and acoustics, shows as clearly as any experiment can show anything, that the source, cause, or origin is common both to the properties of light and the formation of the lenses and retina in the eye—both to the properties of sound and the tympanum, malleus, incus, and stapes of the ear. No doubt whatever can exist upon this subject, any more than, if we saw a particular order issued to a body of men to perform certain uncommon evolutions, and afterwards saw the same body performing those same evolutions, we could doubt their having received the order. A designing and intel-

* An inquiry or discussion of limits, or a limitary investigation in mathematics, is where we seek to know within what limits a solution must be found, as it were, whereabouts.

ligent and skilful author of these admirably adapted works is equally a clear inference from the same facts. We can no more doubt it than we can question, when we see a mill grinding corn into flour, that the machinery was made by some one who designed by means of it to prepare the materials of bread. The same conclusions are drawn in a vast variety of other instances, both with respect to the parts of human and other bodies, and with respect to most of the other arrangements of nature. Similar conclusions are also drawn from our consciousness, and the knowledge which it gives us of the structure of the mind. Thus we find that attention quickens memory and enables us to recollect; and that habit renders all exertions and all acquisitions easy, beside having the effect of alleviating pain.

But when we carry our survey into other parts, whether of the natural or moral system, we cannot discover any design at all. We frequently perceive structures the use of which we know nothing about; parts of the animal frame that apparently have no functions to perform—nay that are the source of pain without yielding any perceptible advantage; arrangements and movements of bodies which are of one particular kind, and yet we are quite at a loss to discern any reason why they might not have been of many other descriptions; operations of nature that seem to serve no purpose whatever; and other operations and other arrangements, chosen equally without any beneficial view, and yet which often give rise to much apparent confusion and mischief. Now, the question is, *first,* whether in any one of these cases of arrangement and structures with no visible object at all, we can for a moment suppose that there really is no object answered, or only conceive that we have been unable to discover it? *Secondly,* whether in the cases where mischief sometimes is perceived, and no other purpose appears to be effected, we do not almost as uniformly lay the blame on our own ignorance, and conclude, not that the arrangement was made without any design, and that mischief arises without any contriver, but that if we knew the whole case we should find a design and contrivance, and also that the apparent mischief would sink into the general good? It is not necessary to admit, for our present purpose, this latter proposition, though it brings us closer to the matter in hand; it is sufficient for the present to admit, what no one doubts, that when a part of the body, for instance, is discovered, to which, like the spleen, we cannot assign any function in the animal system, we never think of concluding that it is made for no use, but only that we have as yet not been able to discover its use.

Now let us ask, why do we arrive, and without any hesitation whatever, or any exception whatever, always and immediately arrive, at this conclusion respecting intelligence and design? Nothing could be more unphilosophical, nay more groundless, than such a process of reasoning, if we had only been able to trace design in one or two instances; for instance, if we found only the eye to show proofs of contrivance, it would be wholly gratuitous, when we saw the ear, to assume that it was adapted to the nature of sound, and still more so, if, on examination, we perceived it bore no perceptible relation to the laws of acoustics. The proof of contrivance in one particular is nothing like a proof, nay does not even furnish the least presumption of contrivance in other particulars; because, *à priori*, it is just as easy to suppose one part of nature to be designed for a purpose, and another part, nay all other parts, to be formed at random and without any contrivance, as to suppose that the formation of the whole is governed by design. Why, then, do we, invariably and undoubtingly, adopt the course of reasoning which has been mentioned, and never for a moment suspect anything to be formed without some reason—some rational purpose? The only ground of this belief is, that we have been able distinctly to trace design in so vast a majority of cases as leaves us no power of doubting that, if our faculties had been sufficiently powerful, or our investigation sufficiently diligent, we should also have been able to trace it in those comparatively few instances respecting which we still are in the dark.

It may be worth while to give a few instances of the ignorance in which we once were of design in some important arrangements of nature, and of the knowledge which we now possess to show the purpose of their formation. Before Sir Isaac Newton's optical discoveries we could not tell why the structure of the eye was so complex, and why several lenses and humours were required to form a picture of objects upon the retina. Indeed, until Dolland's subsequent discovery of the achromatic effect of combining various glasses, and Mr. Blair's still more recent experiments on the powers of different refracting media, we were not able distinctly to perceive the operation and use of the complicacy in the structure of the eye. We now well understand its nature, and are able to comprehend how that which had at one time, nay for ages, seemed to be an unnecessary complexity, forms the most perfect of all optical instruments, and according to the most certain laws of refraction and of dispersion.

So, too, we had observed for some centuries the forms of the orbits in which the heavenly bodies move, and we had found these to be ellipses with a very small eccentricity. But why this was the form of those orbits no one could even conjecture. If any person, the most deeply skilled in mathematical science, and the most internally convinced of the universal prevalence of design and contrivance in the structure of the universe, had been asked what reason there was for the planets moving in ellipses so nearly approaching to circles, he could not have given any good reason, at least beyond a guess. The force of gravitation, even admitting that to be as it were a condition of the creation of matter, would have made those bodies revolve in ellipses of any degree of eccentricity just as well, provided the angle and the force of projection had been varied. Then why was this form rather than any other chosen? No one knew; yet no one doubted that there was ample reason for it. Accordingly the sublime discoveries of Lagrange and La Place ha shown us that this small eccentricity is one material element in the formula by which it is shown that all the irregularities of the system are periodical, and that the deviation never can exceed a certain amount on either hand.

But, again, while we were ignorant of this, perhaps the most sublime truth in all science, we were always arguing as if the system had an imperfection, as if the disturbing forces of the different planets and the sun, acting on one another, constantly changed the orbits of each planet, and must, in a course of ages, work the destruction of the whole planetary arrangement which we had contemplated with so great admiration and with awe. It was deemed enough if we could show that this derangement must be extremely slow, and that, therefore, the system might last for many more ages without requiring an interposition of omnipotent skill to preserve it by rectifying its motions. Thus one of the most celebrated writers above cited argues that, "from the nature of gravitation and the concentricity of the orbits, the irregularities produced are so slowly operated in contracting, dilating, and inclining those orbits, that the system may go on for many thousand years before any extraordinary interference becomes necessary in order to correct it."* And Dr. Burnet adds, that "those small irregularities cast no discredit on the good contrivance of the whole."† Nothing, however, could cast greater discredit if it were as he supposed, and as all men previous to the late discoveries supposed; it was only, they rather think, a "small irregularity," which was every hour tending to the destruction of the whole system, and which must have deranged or confounded its whole structure long before it destroyed it. Yet now we see that the wisdom, to which a thousand years are as one day, not satisfied with constructing a fabric which might last for "many thousand years without His interference," has so formed it that it may thus endure for ever.‡

Now, if such be the grounds of our belief in the universal prevalence of Design, and such the different lights which at different periods of our progress in science we possess upon this great branch of the

* Dr. J. Burnet. Boyle, Lecture ii. c. 78.

† Ib. p. 181.

‡ If the retardation of Encke's comet, or other facts, should lead to the belief of general derangement from an Ether, who will now be bold enough to doubt that further discovery may show the adjustment of this also?

divine government; if we undoubtingly believe that contrivance is universal only because we can trace and comprehend it in a great majority of instances, and if the number of exceptions to the rule is occasionally diminished as our knowledge of the particulars is from time to time extended—may we not apply the same principle to the apprehension of Benevolent purpose, and infer from the number of instances in which we plainly perceive a good intention, that if we were better acquainted with those cases in which a contrary intention is now apparent, we should there too find the generally pervading character of Benevolence to prevail? Not only is this the manner in which we reason respecting the Design of the Creator from examining his works; it is the manner in which we treat the conduct of our fellow-creatures. A man of the most extensive benevolence and strictest integrity in his general deportment has done something equivocal; nay, something apparently harsh and cruel; we are slow to condemn him; we give him credit for acting with a good motive and for a righteous purpose; we rest satisfied that "if we only knew everything, he would come out blameless." This arises from a just and a sound view of human character, and its general consistency with itself. The same reasoning may surely be applied, with all humility and reverence, to the works and the intentions of the great Being who has implanted in our mind the principles which lead to that just and sound view of the deeds and motives of men.

But let the argument be rested upon our course of reasoning respecting divine contrivance. The existence of Evil is in no case more apparent than the existence of Disorder seems to be in many things. To go no further than the last example which has been given—the mathematician could perceive the derangement in the planetary orbits, could demonstrate that it must ensue from the mutual action of the heavenly bodies on each other, could calculate its progress with the utmost exactness, could tell with all nicety how much it would alter the forms of the orbits in a given time, could foresee the time when the whole system must be irretrievably destroyed by its operation as a mathematical certainty. Nothing that we call evil can be much more certainly perceived than this derangement, of itself an evil, certainly a great imperfection, if the system was observed by the mind of man as we regard human works. Yet we now find, from well considering some things which had escaped attention, that the system is absolutely free from derangement; that all the disturbances counterbalance each other; and that the orbits never can either be flattened or bulged out beyond a definite and a very inconsiderable quantity. Can any one doubt that there is also a reason for even this small and limited, this regular and temporary, derangement? Why it exists at all, or in any the least degree, we as yet know not. But who will presume to doubt that it has a reason which would at once satisfy our minds were it known to us? Nay, who will affirm that the discovery of it may not yet be in reserve for some later, and happier age? Then are we not entitled to apply the same reasoning to what at present appears Evil in a system of which, after all we know of it, so much still remains concealed from our view?

The mere act of creation in a Being of wisdom so admirable and power so vast, seems to make it extremely probable that perfect goodness accompanies the exertion of his perfect skill. There is something so repugnant to all our feelings, but also to all the conceptions of our reason, in the supposition of such a Being desiring the misery, for its own sake, of the beings whom he voluntarily called into existence and endowed with a sentient nature, that the mind naturally and irresistibly recoils from such a thought. But this is not all. If the nature of that great Being were evil, His power being unbounded, there would be some proportion between the amount of ills and the monuments of that power. Yet we are struck dumb with the immensity of His works to which no imperfection can be ascribed, and in which no evil can be traced; while the amount of mischief that we see might sink into a most insignificant space, and is such as a being of most inconsiderable power and very limited skill could easily have accomplished. This is not the same consideration with the balance of good against evil; and inquirers do not seem to have sufficiently attended to it. The argument, however, deserves much attention, for it is purely and strictly inductive. The divine nature is shown to be clothed with prodigious power and incomparable wisdom and skill,—power and skill so vast and so exceeding our comprehension, that we ordinarily term them infinite, and are only inclined to conceive the possibility of limiting, by the course of the argument upon evil, one alternative of which is assumed to raise an exception.* But admitting, on account of the question

* The dilemma of the Epicureans, "Aut vult et non potest tollere mala."

under discussion, that we have only a right to say the power and skill are prodigiously great, though possibly not boundless, they are plainly shown in the phenomena of the universe to be the attributes of a Being, who, if evil-disposed, could have made the monuments of Ill upon a scale resembling those of Power and Skill; so that if those things which seem to us Evil be really the result of a mischievous design in such a Being, we cannot comprehend why they are upon so entirely different a scale. This is a strong presumption from the facts that we are wrong in imputing those appearances to such a disposition. If so, what seems evil must needs be capable of some other explanation, consistent with divine goodness,—that is to say, would not prove to be evil at all, if we knew the whole of those facts.

But it is necessary to proceed a step further, especially with a view to the fundamental position now contended for, the extending to the question of Benevolence the same principles which we apply to that of Intelligence. The Evil which exists, or that which we suppose to be Evil, not only is of a kind and a magnitude requiring inconceivably less power and less skill than the admitted good of the creation —it also bears a very small proportion in amount;

quite as small a proportion as the cases of unknown or undiscoverable design bear to those of acknowledged and proved contrivance. Generally speaking, the preservation and the happiness of sensitive creatures appears to be the great object of creative exertion and conservative providence. The expanding of our faculties, both bodily and mental, is accompanied with pleasure; the exercise of those powers is almost always attended with gratification; all labour so acts as to make rest peculiarly delicious; much of labour is enjoyment; the gratification of those appetites by which both the individual is preserved and the race is continued, is highly pleasurable to all animals; and it must be observed that instead of being attracted by grateful sensations to do anything requisite for our good or even our existence, we might have been just as certainly urged by the feeling of pain, or the dread of it, which is a kind of suffering in itself. Nature, then, resembles the lawgiver who, to make his subjects obey, should prefer holding out rewards for compliance with his commands rather than denounce punishments for disobedience. But nature is yet more kind; she is gratuitously kind; she not only prefers inducement to threat or compulsion, but she adds more gratification than was necessary to make us obey her calls. How well might all creation have existed and been continued, though the air had not been balmy in spring, or the shade and the spring refreshing in summer; had the earth not been enamelled with flowers, and the air scented with perfumes! How needless for the propagation of plants was it that the seed should be enveloped in fruits the most savoury to our palate, and if those fruits serve some other purpose, how foreign to that purpose was the formation of our nerves so framed as to be soothed or excited by their flavour! We here perceive Design, because we trace adaptation. But we at the same time perceive Benevolent Design, because we perceive gratuitous and supererogatory enjoyment bestowed. Thus, too, see the care with which animals of all kinds are tended from their birth. The mother's instinct is not more certainly the means of securing and providing for her young, than her gratification in the act of maternal care is great and is also needless for making her perform that duty. The grove is not made vocal during pairing and incubation, in order to secure the laying or the hatching of eggs; for if it were as still as the grave, or were filled with the most discordant croaking, the process would be as well performed. So, too, mark the care with which injuries are remedied by what has been correctly called the *vis medicatrix*. Is a muscle injured?—Suppuration takes place, the process of granulation succeeds, and new flesh is formed to supply the gap, or if that is less wide, a more simple healing process knits together the severed parts. Is a bone injured?—A process commences by which an extraordinary secretion of bony matter takes place, and the void is supplied. Nay, the irreparable injury of a joint gives rise to the formation of a new hinge, by which the same functions may be not inconveniently, though less perfectly, performed. Thus, too, recovery of vigour after sickness is provided for by increased appetite; but there is here superadded, generally, a feeling of comfort and lightness, an enjoyment of existence so delightful, that it is a common remark how nearly this compensates the sufferings of the illness. In the economy of the mind it is the same thing. All our exertions are stimulated by curiosity, and the gratification is extreme of satisfying it. But it might have been otherwise ordered, and some painful feeling might have been made the only stimulant to the acquisition of knowledge. So, the charm of novelty is proverbial; but it might have been the unceasing cause of the most painful alarms. Habit renders every thing easy; but the repetition might have only increased the annoyance. The loss of one organ makes the others more acute. But the partial injury might have caused, as it were, a general paralysis. 'Tis thus that Paley is well justified in exclaiming, "It is a happy world after all!" The pains and the sufferings, bodily and mental, to which we are exposed, if they do not sink into nothing, at least retreat within comparatively narrow bounds; the ills are hardly seen when we survey the great and splendid picture of worldly enjoyment or ease.

But the existence of considerable misery is undeniable; and the question is, of course, confined to that. Its exaggeration, in the ordinary estimate both of the vulgar and of sceptical reasoners, is equally certain. Paley, Bishop Sumner, as well as Derham, King, Ray, and others of the older writers, have made many judicious and generally correct observations upon its amount, and they, as well as some of the able and learned authors of the Bridgewater Treatises, have done much in establishing deductions necessary to be made, in order that we may arrive at the true amount. That many things, apparently unmixed evils, when examined more narrowly, prove to be partially beneficial, is the fair result of their well-meant labours; and this, although anything rather than a proof that there is no Evil at all, yet is valuable as still further proving the analogy between this branch of the argument and that upon Design; and in giving hopes that all may possibly be found hereafter to be good, as everything will assuredly be found to be contrived with an intelligent and useful purpose. It may be right to add a remark or two upon some evils, and those of the greatest magnitude in the common estimate of human happiness, with a view of further illustrating this part of the subject.

Mere Imperfection must altogether be deducted from the account. It never can be contended that any evil nature can be ascribed to the first cause, merely for not having endowed sentient creatures with greater power or wisdom, for not having increased and multiplied the sources of enjoyment, or for not having made those pleasures which we have more exquisitely grateful. No one can be so foolish as to argue that the Deity is either limited in power, or deficient in goodness, because he has chosen to create some beings of a less perfect order than others. The mere negation in the creating of some, indeed of many, nay of any conceivable number of desirable attributes, is therefore no proper evidence of evil design or of limited power in the Creator—it is no proof of the existence of Evil properly so called. But does not this also erase death from the catalogue of ills? It might well please the Deity to create a mortal being—a being which, consisting of soul and body, was only to live upon this earth for a limited number of years. If, when that time has expired, this being is removed to another and a superior state of existence, no evil whatever accrues to it from the change; and all views of the government of this world lead to the important and consolotary conclusion, that such is the design of the Creator; that he cannot have bestowed on us minds capable of such expansion and culture only to be extinguished when they have reached their highest pitch of improvement; or if this be considered as begging the question by assuming benevolent design, we cannot easily conceive that while the mind's force is so little affected by the body's decay, the destruction or dissolution of the latter should be the extinction of the former.* But that death operates as an evil of the very highest kind in two ways is obvious; the dread of it often embitters life, and the death of friends brings to the mind by far its most painful infliction; certainly the greatest suffering it can undergo without any criminal consciousness of its own.

For this evil, then—this grievous and admitted evil—how shall we account? But first let us consider whether it be not unavoidable; not merely under the present dispensation, and in the existing state of things; for that is wholly irrelevant to the question which is raised upon the fitness of this very state of things; but whether it be not a necessary evil. That man might have been created immortal is not denied; but if it were the will of the Deity to form a limited being and to place him upon the earth for only a certain period of time, his death was the necessary consequence of this determination. Then as to the pain which one person's removal inflicts upon surviving parties, this seems the equally necessary consequence of their having affections. For if any being feels love towards another, this implies his

* A Note is subjoined on the Resurrection of the Body.

desire that the intercourse with that other should continue; or, which is the same thing, the repugnance and aversion to its ceasing; that is, he must suffer affliction for that removal of the beloved object. To create sentient beings devoid of all feelings of affection was no doubt possible to Omnipotence; but to endow those beings with such feelings as should give the constant gratification derived from the benevolent affections, and yet to make them wholly indifferent to the loss of the objects of those affections, was not possible even for Omnipotence; because it was a contradiction in terms, equivalent to making a thing both exist and not exist at one and the same time. Would there have been any considerable happiness in a life stripped of these kindly affections? We cannot affirm that there would not, because we are ignorant what other enjoyments might have been substituted for the indulgence of them. But neither can we affirm that any such substitution could have been found; and it lies upon those who deny the necessary connexion between the human mind, or any sentient being's mind, and grief for the loss of friends, to show that there are other enjoyments which could furnish an equivalent to the gratification derived from the benevolent feelings. The question then reduces itself to this: Wherefore did a Being, who could have made sentient beings immortal, choose to make them mortal? or, Wherefore has he placed man upon the earth for a time only? or, Wherefore has he set bounds to the powers and capacities which he has been pleased to bestow upon his creature? and this is a question which we certainly never shall be able to solve; but a question extremely different from the one more usually put—How happens it that a good Being has made a world full of misery and death?

In the necessary ignorance wherein we are of the whole designs of the Deity, we cannot wonder if some things, nay if many things, are to our faculties inscrutable. But we assuredly have no right to say that those difficulties which try and vex us are incapable of a solution, any more than we have to say, that those cases in which as yet we can see no trace of design, are not equally the result of intelligence, and equally conducive to a fixed and useful purpose with those in which we have been able to perceive the whole, or nearly the whole, scheme. Great as have been our achievements in physical astronomy, we are as yet wholly unable to understand why a power pervades the system acting inversely as the squares of the distance from the point to which it attracts, rather than a power acting according to any other law; and why it has been the pleasure of the almighty Architect of that universe, that the orbits of the planets should be nearly circular instead of approaching to, or being exactly the same with many other trajectories of a nearly similar form, though of other properties; nay, instead of being curves of a wholly different class and shape. Yet we never doubt that there was a reason for this choice; nay, we fancy it possible that even on earth we may hereafter understand it more clearly than we now do; and never question that in another state of being we may be permitted to enjoy the contemplation of it. Why should we doubt that, at least in that higher state, we may also be enabled to perceive such an arrangement as shall make evil wholly disappear from our present system, by showing us that it was necessary and inevitable, even in the works of the Deity; or, which is the same thing, that its existence conduces to such a degree of perfection and of happiness upon the whole, as could not, even by Omnipotence, be attained without it; or, which is also the same thing, that the whole creation as it exists, taking both worlds together, is perfect, and incapable of being in any particular changed without being made worse and less perfect?

Taking both worlds together—For certainly were our views limited to the present sublunary state, we may well affirm that no solution whatever could even be imagined of the difficulty—If we are never again to live; if those we here loved are for ever lost to us; if our faculties can receive no further expansion; if our mental powers are only trained and improved to be extinguished at their acme—then indeed are we reduced to the melancholy and gloomy dilemma of the Epicureans; and Evil is confest to checker, nay almost to cloud over, our whole lot, without the possibility of comprehending why, or of reconciling its existence with the supposition of a Providence at once powerful and good. But this inference is also an additional argument for a future state, when we couple it with those other conclusions respecting the economy of the world to which we are led by wholly different routes, when we investigate the phenomena around us and within us.

Suppose, for example, it should be found that there are certain purposes which can in no way whatever—no conceivable way—be answered except by placing man in a state of trial or probation;

suppose the essential nature of mind shall be found to be such, that it could not in any way whatever exist so as to be capable of the greatest purity and improvement—in other words, the highest perfection—without having undergone a probation; or suppose it should be found impossible to communicate certain enjoyments to rational and sentient beings without having previously subjected them to certain trials and certain sufferings—as for instance, the pleasures derived from a consciousness of perfect security, the certainty that we can suffer and perish no more—this surely is a possible supposition. Now, to continue the last example—Whatever pleasure there is in the contrast between ease and previous vexation or pain, whatever enjoyment we derive from the feeling of absolute security after the vexation and uncertainty of a precarious state, implies a previous suffering—a previous state of precarious enjoyment; and not only implies it but necessarily implies it, so that the power of Omnipotence itself could not convey to us the enjoyment without having given us the previous suffering. Then is it not possible that the object of an all powerful and perfectly benevolent Being should be to create like beings, to whom as entire happiness, as complete and perfect enjoyment, should be given as any created beings—that is, any being, except the Creator Himself—can by possibility enjoy? This is certainly not only a very possible supposition, but it appears to be quite consistent with, if it be not a necessary consequence of, His being perfectly good as well as powerful and wise. Now we have shown, therefore, that such being supposed the design of Providence, even Omnipotence itself could not accomplish this design, as far as one great and important class of enjoyments is concerned, without the previous existence of some pain, some misery. Whatever gratification arises from relief—from contrast—from security succeeding anxiety — from restoration of lost affections—from renewing severed connexions—and many others of a like kind, could not by any possibility be enjoyed unless the correlative suffering had first been undergone. Nor will the argument be at all impeached by observing, that one Being may be made to feel the pleasure of ease and security by seeing others subjected to suffering and distress; for that assumes the infliction of misery on those others; it is "*alterius* spectare laborem" that we are supposing to be sweet; and this is still partial evil.

As the whole argument respecting evil must, from the nature of the question, resolve itself into either a proof of some absolute or mathematical necessity not to be removed by infinite power, or the showing that some such proof may be possible although we have not yet discovered it,* an illustration may naturally be expected to be attainable from mathematical considerations. Thus we have already adverted to the law of periodical† irregularities in the solar system. Any one before it was discovered seemed entitled to expatiate upon the operation of the disturbing forces arising from mutual attraction, and to charge the system arranged upon the principle of universal gravitation with want of skill, nay, with leading to inevitable mischief,—mischief or evil of so prodigious an extent as to exceed incalculably all the instances of evil and of suffering which we see around us in this single planet. Nevertheless what then appeared so clearly to be a defect and an evil, is now well known to be the very absolute perfection of the whole heavenly architecture.

Again, we may derive a similar illustration from

* This proposition has been overlooked by many reasoners, as we have seen above; yet is manifestly true.

† These derangements are also called *secular* to distinguish them from others which are termed periodical. But in the view of our argument both are of a periodical kind.

a much more limited instance, but one immediately connected with strict mathematical reasoning, and founded altogether in the nature of necessary truth. The problem has been solved by mathematicians, Sir Isaac Newton having first investigated it, of finding the form of a symmetrical solid, or solid of revolution, which in moving through a fluid shall experience the least possible resistance; in other words, of finding the form that must be impressed upon any given bulk of matter, so that it shall move more easily through a surrounding fluid than if it had any other conceivable form whatever, with a breadth or a length also given. The figure bears a striking resemblance to that of a fish. Now suppose a fish were formed exactly in this shape, and that some animal endowed with reason were placed upon a portion of its surface, and able to trace its form for only a limited extent, say at the narrow part, where the broad portion or end of the moving body was opposed, or seemed as if it were opposed, to the surrounding fluid when the fish moved—the reasoner would at once conclude that the contrivance of the fish's form was very inconvenient and inartificial, and that nothing could be much worse adapted for expeditious or easy movement through the waters. Yet it is certain that upon being afterwards permitted to view THE WHOLE body of the fish, what had seemed a defect and an evil, not only would appear plainly to be none at all, but it would appear manifest that this seeming evil or defect was a part of the most perfect and excellent structure, which it was possible even for Omnipotence and Omniscience to have adopted, and that no other conceivable arrangement could by possibility have produced so much advantage, or tended so much to fulfil the design in view. Previous to being enlightened by such an enlarged view of the whole facts, it would thus be a rash and unphilosophical thing in the reasoner whose existence we are supposing to pronounce an unfavourable opinion. Still more unwise would it be if numerous other observations had evinced traces of skill and goodness in the fish's structure. The true and the safe conclusion would be to suspend an opinion which could only be unsatisfactorily formed upon imperfect data; and to rest in the humble hope and belief that one day all would appear for the best.*

* See further illustrations of these remarks under the head of *Researches on Fossil Osteology*, sub fin.

OF

CONFLICTING INSTINCTS

AND CONFLICTING

CONTRIVANCES GENERALLY.

This subject is deserving of attention, because the facts are curious, and the appearances are attended with much difficulty to the theological inquirer. But it belongs properly to the general head of evil, or apparent evil and imperfection. Nor is there any reason to expect that we shall ever, in the present limited state of our faculties, and while placed in a narrow and bounded state of existence, be able to penetrate the obscurity which surrounds the question upon every side.

When the expression *conflicting instincts* is employed, it does not so much denote instincts conflicting in the same nature, for example in the same animal or tribe of animals, as that the instinct with which one is endowed appears to be given it for the purpose of counteracting, thwarting, frustrating, or evading some other instinct bestowed upon another animal. Thus the contrivances by which one tribe endeavours to seize another and make prey of it are in conflict with those by which the latter endeavour to defend themselves or to escape. The sepia, or cuttle-fish, voids a black liquid, which prevents another fish from finding it or continuing its pursuit. The woodpecker is led to strike its long bill violently into branches where certain insects lie instinctively concealing themselves to escape destruction. Certain birds build their nests so as to avoid the reptiles which swarm around, and have the natural appetite to feed upon those birds. This, however, may be referred to reasoning on the one hand and mere appetite on the other. But animals that prey upon others are led by undoubted instinct to seek the places where they are to be found, and to breed at the seasons when they are, in consequence of other instincts, produced, so as to become the food of the former. Some animals seem made, or at least provided, with appetites on purpose to devour the embryos of others or prevent their increase. Thus, some fish feed on the spawn of others, and the ichneumon feeds upon the eggs of the crocodile.

Akin to such conflicting instincts, or rather forming the physical, as these compose the mental, class, are contrivances of a kind apparently designed one to counteract the other. This is particularly observable in the structure of animals as taken in connexion with their habits. Thus, some beasts of prey are formed for running down, some for springing upon, other animals, which, on their part, are provided with forms that favour their escape. The lion and tiger have vertebræ connected with their ribs and with each other, so as to facilitate by a lateral mobility their crawling and leaping. On the other hand, the spine of deer and hares, and other defenceless animals, have the vertebræ so contrived as to facilitate their escape, and the eyes so placed as to warn them of attacks from behind, and from the sides, as well as in front. The serpent's backbone is a singular and a beautiful structure. It has three or four times the usual number of joints, and they play on one another like ball and socket. The poison too of the few venomous species is curiously secreted in a bag placed beneath a moveable tooth, which is perforated with a tube or duct that terminates in the poison sac, and is continued to the sharp point on the other end, so that when the animal bites, the tooth, pressing on the sac, makes the poison squirt through the duct of the tooth into the wound made by its point. No more striking proof of design can be given than this. Then the rattle in the tail of the most deadly of the tribe gives warning to keep out of its way, and thus as it were prevent the machinery of destruction from being of any use to the animal, unless, perhaps, as a weapon of defence, when he is attacked by some one that disregards the warning. Again, birds are furnished with a defence or shield to protect their eyes in flying through the thickets. They are also furnished with a power of contracting their eyes, so as to adjust them to the distances of various objects. But birds of prey have a peculiar mechanism for this purpose. Their eye is provided with a kind of muscle, loop like, which enables them to compress the lens so as to adjust it for descrying objects at a vast distance, acting like the slide of a telescope, and used to effect the same purpose, that is to suit the focus of the eye. Now this can be of no use excepting as a means of attack and destruction; for the adjustment to near distances can alone help the animal to defend itself. On the other hand, weak birds are furnished with many important means of escaping from their more powerful enemies. Similar observations may be made upon the structure and habits of fishes. Thus the sword-fish is provided with a most powerful weapon and with great muscular strength to use it. He attacks the whale, which immediately, and by a special instinct, dives into so deep water, that the sword-fish, being wholly unable to bear the pressure, is forced to quit his hold. This pressure produces no inconvenience to the whale, whose structure is formed to bear it with perfect ease.

The *vis medicatrix*, in all its branches, affords striking examples of the same conflict. For while the animal body is exposed to injury from its formation, the qualities of its component parts, and the properties of other bodies, and while it is also exposed to injury from disease, there is bestowed upon it a power of not only resisting and avoiding those injuries, but also of repairing them after they have been inflicted. Thus the eye is so fixed in the socket and so protected by the eyelashes and the eyebrow, that irritating particles do not easily reach its tender surface, or perhaps we should rather say, the tender and sensitive parts surrounding it. But if anything does fix upon it, there is provided a sudden and copious secretion of watery fluid, which sheathes the parts attacked, and tends to wash away or expel the foreign substance that has intruded itself. So new bone and new flesh are produced to supply any void made by accidents, and to make the severed parts knit again and heal. In like manner, when an extraneous substance has, by the laws of matter and motion, been introduced into any limb, or into the cavities of the body, and cannot be removed, a new formation of flesh, or cartilage, or bone, according to the place where it is imbedded, takes place, and covers it over, so as to defend the adjacent parts and enable the system to be continued in its operations. Again, where a disease attacks the frame according to the properties of the system on the one hand, and the qualities of infection or other noxious effluvia on the other hand, the system is thrown into a state which produces a sudden and often violent effort to throw off the mischief; or if this fails, then other efforts are made to resist and to remedy the damage sustained, and to restore health.*

All these conflicts and inconsistencies belong to

* The illustrations of design drawn from the operation of the *Vis Medicatrix* form the subject of a separate note.

the general head of imperfection, or of evil. They are merely other cases of what we have specified with various views in the argument upon that great question. They fall altogether within the scope of that argument; therefore they require no separate discussion in this place. It is quite evident that we have no more right to regard a conflict of contrivances as any real inconsistency in reference to the whole design which is concealed from us, than we have to regard any part as formed without meaning and use, because we have not discovered its use. We cannot say that one part of a machine counteracts another part, unless we can perceive distinctly what the purpose of the whole mechanism is. The apparent opposition may be necessary for accomplishing that purpose: as the friction of one wheel upon another is necessary to the action of both, or the counteraction of one lever by another to the motion of the whole.

DOCTRINE OF UBIQUITY.

Two opinions have principally been held upon this subject, and have divided metaphysical theologians, though without giving rise to any very great vehemence of controversy. The one class have maintained that the Deity is everywhere present in His person, substance, or essence; this is termed the doctrine of *Essential* Ubiquity, to distinguish it from that of *Virtual* Ubiquity, held by the other class, who have maintained that presence in place cannot be predicated of mind at all, still less of the infinite mind, which only acts everywhere by its power. A very obscure and imperfect notion has prevailed with some, chiefly of the ancient sects, as if universal matter or infinite space were constituted by the Deity's essence; but this would plainly make Him divisible, and indeed material. The ancient idea of His being the universal soul, the *anima mundi*, related to the world as the soul to the body, is equally unfounded, though approaching more to Essential Ubiquity.

The opinions of the ancients upon this great and difficult question were not, perhaps, materially different from the first of these doctrines. According to Cicero (*De Nat. Deor,* lib. i.), Pythagoras taught "Deum esse animum per naturam rerum omnium intentum et commeantem." And in his treatise *De Legg.* (lib. ii.) he says that Thales of Miletus first laid down the well-known position, "Deorum omnia esse plena." The passage in Seneca (*De Benef.* lib. iv.) is also worthy of notice, as containing the doctrine in terms: "Quocunque te flexeris ibi Deum videbis occurrentem tibi. Nihil ab illo vacat. Opus suum ipse implet." And again in *Epist.* 15, "Ubique et omnibus preest."

Among the moderns, the followers of Socinus are those who have most strenuously denied the essential ubiquity; and nothing can be more inconclusive than their reasonings against it. When they urge, for example, that this would degrade the nature of the Deity by supposing Him to inhabit vile and impure places, the answer of Dr. Hancock (Boyle Lecture, II. 222.) is decisive, that this supposes Him of an inferior and animal nature; and that indeed the whole argument savours of anthropomorphitism. They maintain that

He resides in heaven, but that by His power He is felt everywhere, or has a potential and virtual ubiquity only. But first, how have we any right to confine His being, as if He were a human or other finite existence, to one place, and not make the whole universe and all space, even as yet unfilled with any creature, His residence? Secondly, how can we conceive Him preserving and upholding and directing where He is not? It is inconceivable to our minds how power, or any other thing or influence, can act at a distance. It must further be observed that the Socinians, who hold the doctrine of a finite spirit, have quite as great a difficulty to contend with as that imputed by them to the Essential Ubiquity; for theirs is at the least as hard to conceive.

That St. Paul adopted the principle of Essential Ubiquity is evident from what he says both in Acts xvii. and in Heb. i. In the former passage he says, "In Him we live, and move, and have our being;" in the latter, "He upholds all things by the word of His power." So a passage from Jeremiah xxiii., referred to by Sir I. Newton, "Am I a God at hand, saith the Lord, and not a God afar off? Do not I fill heaven and earth?"

The reasoning of Bishop Law upon this subject is certainly by no means satisfactory. It occurs in the seventh note to chapter 1. s. 2; but chiefly in Remark 4 to the third section of that chapter. In answer to Dr. Clarke's position (Reply to Leibnitz), that space is the place of all ideas, the bishop says, that to conceive an immaterial thinking substance in any connexion with the ideas of space is impossible. He adds, that space and spirit are "as distant and incompatible as the most remote and inconsistent things in nature;" and then observes, "that an extended soul seems just such another thing as a green sound, an ell of consciousness, or a cube of virtue." When Dr. Clarke, admitting that extension cannot belong to thought, says that Thought is not Being, the bishop argues against this difference, and, because we only know Being by its thoughts, contends that Being is an aggregate of its properties. When he comes to the question of ubiquity in the Remarks, he says that the notion of the Deity's presence in His simple essence in every part of the boundless immensity, cannot be included in the idea of Omnipresence, because any idea of extension or expansion is inconsistent with that simple essence. But he guards himself against being supposed to maintain that there is any separation of the attributes from the essence, or that his knowledge and power act apart from his essence; nay, he holds that his essence has no more relation to space than those attributes have. He then quotes with commendation the remarks of Episcopius (*Theolog. Inst.*), that the idea of space without matter to fill it is "nihil omninò reale, sed pure pute imaginarium et prorsus nihilum;" and that the very idea of *presence*—of *being in*—implies some reality. And the bishop then goes on to maintain that the Deity knows and acts upon all parts of the universe, as we know from the effects; but that to speak of His acting in extra-mundane space is incomprehensible, and that it is no less so to speak of His actual presence in any part or parts of extension, except it be metaphorically, as eternal truths are said to be the same in all times and places, though they really have no relation to either. In like manner he disposes of the position, that nothing can act where it is not, as applied to divine power, by urging that this is still supposing spirit to exist somewhere or to be circumscribed by some parts of space, contrary, as he thinks, to the very nature of spirit as distinguished from matter.

It must be allowed that this is by no means a satisfactory statement of the question, and is very far from disproving the Essential Ubiquity, which it seems to deny, although not, perhaps, directly or in very express terms. The assumption pervades the whole reasoning, that because a finite mind can have no relation to matter in its essence, but is only united with a being, therefore the infinite mind cannot also comprehend all matter, or at least unite within itself any given number of material qualities, as well as all mental qualities, in their perfection. Nor is it shown to be at all inconsistent with the nature of such a mind, that it may not exist essentially as well as virtually or potentially where no matter is. Episcopius's argument, founded on the necessity of presence, as if it was existing or being *in* something, is plainly erroneous, for it supposes that when any being or any mind is affirmed to be present in any place, it is therefore bounded by, or attached to the things in that place; whereas the doctrine of Essential Ubiquity assumes that, although present in any given place, the Deity is also and at the same time not confined to that place, but present everywhere else. Then to speak of Essential Ubiquity as incomprehensible, proves little, unless Virtual Ubiquity be shown to be less incomprehensible. The subject is altogether of so high and obscure a nature, that whoever ventures to contemplate it, must admit the inadequacy of human reason or human imagination to form any clear ideas respecting it. But a Being acting where he is not seems of all notions the least conceivable; indeed we seem to be rather the dupes of language in so speaking, and to be putting together words without any very distinct meaning annexed to them.

The greatest force of authority, in weight as well as in number, is certainly in favour of Essential Ubiquity. Some philosophers, as Descartes, have overlooked the subject altogether; others, as Paley, have considered that natural theology is silent upon the subject. Sir Isaac Newton peremptorily, and in terms of peculiar eloquence and force, declares for Essential Ubiquity, in that part of the celebrated general scholium, in which he sums up the divine qualities and Omnipresence among the rest (contrary to Paley's hasty assertion), as legitimate inferences from the phenomena of nature, and a branch of natural philosophy. " Deus est unus et idem Deus semper et ubique. Omnipræsens est, non per *virtutem* solam, sed etiam per *substantiam*: nam virtus sine substantiâ subsistere non potest. In ipso continentur et moventur universa, sed sine mutuâ passione. Deus nihil patitur ex corporum motibus; illa nullam sentiunt resistentiam ex Omnipræsentiâ Dei. Deum summum necessario existere in confesso est; et eâdem necessitate *semper* est et *ubique*." The whole of this great passage concludes thus, "Et hæc de Deo, de quo utique ex phenomenis disserere, ad philosophiam naturalem pertinet."—(Principia, lib. iii. Schol. Gen. *sub fin.*

NOTE UPON THE RESURRECTION.

Many arguments have been held upon this subject both in ancient and modern times; but chiefly in modern; because the ancient sects, generally speaking, held that a future state consisted in the immortality of the soul only, and that matter was essentially and necessarily subject to decay and dissolution. Indeed, they for the most part limited the Deity's power to moulding and moving matter, but conceived that matter was eternal like the Deity, and that it had certain fundamental qualities, as corruptibility, which no power could alter. Moreover, many of those philosophers held that all minds being emanations from the divine mind, would ultimately be reunited with it. So that the surviving in any state of individual existence, or what we should consider as any real immortality of the soul, any true future state of beings who had lived in this world, could hardly be said to be the belief of those inquirers.* Others, no doubt, believed in such a future state as approaches nearer to our own ideas; but very few considered the Resurrection of the Body as the mode in which our existence is to be continued.

Among these few appear to have been the Stoics; for Lactantius cites a passage from Chrysippus, which shows that they considered it clear that there was nothing impossible in the resurrection of the body—*δήλον ὡς οὐδεν ἀδύνατον ἀπκαταναστησεσθαι.* The precision with which the Christian revelation dwells upon this appears to have been rendered highly desirable, if not necessary, by the prevalence of the notion among such as did not doubt of a future state, that it was to consist only of a reunion with the essence of the Deity, or a confusion with the *anima mundi,* as some expressed it. But

* It is worthy of remark that the ancient doctrine of Emanation from the Deity and reunion with Him, is the belief of the rudest and simplest, as it was of the most civilized men. The inhabitants of the South Sea Islands believe in a future state of this kind; and the only punishment which their self-indulgent notions recognise is, that which impurity in this world may make necessary to purify the soul before it is absorbed in the divine essence. This union is, however, according to their ideas, only temporary; the Deity is afterwards to give them a station in eternal twilight, or in eternal night, according to their conduct and nature. (*Cook's Second Voyage*, i. 164.)

some learned Christians have maintained that the gospel only states the immortality of the soul, and not the rising and immortality of the body; particularly a reverend author (Mr. Bourne) of a treatise upon the true nature of the Christian ἀναστασις, published about eighty years ago.

The difficulties thrown by sceptics in the way of the Christian doctrine, as commonly received, have given rise to this and to other theories,—as the speculations of Grotius,—but these objections really appear to be without any solid foundation, although at first sight plausible enough.

When it is said that the same matter going successively to form a great many human bodies, the divine or infinite power itself never can make all men rise with the same bodies, because it is an absolute impossibility that any one mass of matter should be in two places, or belong to two souls, at the same time; it is assumed that every one particle of each body is to be reunited to every other, and the whole body to the soul. Now, not only is this a merely gratuitous assumption—it is plain upon the least consideration, that it is contrary, and necessarily contrary, to the very idea of the Resurrection. For the particles of any given body at the time of its death are not at all those of the same body two or three years before; nor are those which composed it three years before the same with those which composed it six; and thus, if the words Resurrection of the Body be taken literally, not only must each person rise as he died, but rise with twenty or thirty bodies in one, which is manifestly absurd.

The Christian doctrine is, that matter shall be united with mind in a future state as it has been in this, or at least, that the mind shall revive or be continued with matter—with some body—though how long the union is to endure is nowhere said. It is also a part of the Christian doctrine, that the Deity created matter and can mould as well as create. Therefore a single particle of the former body could be just as easily formed by divine power into a whole body resembling the one last united to the soul on earth, as He can raise that body or continue the existence of the soul. Nor will it only be similarity—there will be identity; for personal identity does not at all depend upon the proportion of particles which remain united with each other; else no individual could feel and believe that he was the same one year as another. But unless this or some such view as this be taken of the subject, the objection becomes irresistible. Thoughtless and zealous persons have sometimes fancied they could overcome it by saying, that with the Deity all things are possible, or, which is the same thing, that by working a miracle, He can give each soul exactly its former body. But those things only are possible which involve no contradictions; and it is as utter a contradiction in terms and in ideas to suppose the same particles belonging to different bodies and different souls at one and the same time, as to suppose that the whole can be greater than the sum of all its parts. So, a miracle means the suspension of the laws of nature, or a deviation from those rules prescribed by the divine power. But the giving the same particles to different bodies and souls at the same time is not suspending the laws of nature, but altering the truths of mathematics, which are necessary, clear, and indisputable. In short, the slightest attention to the subject must show that the sceptical objection, though futile enough when duly considered and met by the appropriate answer, never can be removed by any such answer as those unreflecting persons fancy they are giving it, when they only affirm the extent of infinite power.

The most learned and most orthodox divines, accordingly, have taken views somewhat similar to those here unfolded. Thus Dr. Ibbot, in his sermons (Boyle Lecture, ii.), lays it down as clear that there is no kind of necessity for being able to determine whether the raised body shall consist of the same particles as were laid in the grave, or the same several particles which were united to the soul during life, or of particles not so united, or whether the soul shall not have a body consisting of only particles of matter indifferently. These differences he holds not to be fundamental; and this he holds because personal identity, the grand point, depends not on the body but the mind, that is upon memory. But he adds that those who question whether we shall rise with any body at all are no Christians (p. 775, 6). It has already been remarked, however, that some very pious Christians have ventured to call this position in question; although undoubtedly the great weight of authority is against them; and the etymological arguments all tend the same way.

Dr. S. Clarke has argued the question learnedly and ingeniously which Grotius raised, in his celebrated treatise, *De Veritate Religionis Christianæ,*

(ii. 10). Pressed, as it should seem, by the sceptical objection, and assuredly more pressed than he ought to have been, this great writer, hardly less profound in his theological than his juridical works, contended that a constant miracle is going on in the human body, to prevent the particles of one frame ever becoming parcel of another material frame. So that we are called upon to believe against all evidence, as well as all probability, that no particle whatever of fluid, or solid, or gas, which has ever formed a portion of the body of any person, can be assimilated when taken into the stomach of any other person, or indeed of any other animal. Now it is hardly necessary to observe that this hypothesis, unlike most others upon such subjects, is not only wild and extravagant beyond all measure, but is capable of being brought to the test of experiment, and is utterly contrary to the fact. Any animal fed upon the body of a human being will be just as well nourished as upon any other food, and so there can be no doubt would any human being who should indulge in so brutal an act of cannibalism. But the argument of Dr. Clarke is applied to show the groundlessness of the supposed objection which had driven Grotius to frame such a theory. He shows, first, that the real body which is to be raised up may be the original nucleus or stamina, and all the rest merely a superfluous addition of bone, muscles, and fluids, only necessary for earthly purposes. This nucleus, he thinks, may very possibly never undergo any change either during life or at death. Secondly, he says that there may be some seat of the soul which may be material although insensible, some matter of a very refined nature like the seminal principle of plants, or seminal aura, and that this may be unalterable and indissoluble.

We may, however, remark, *first*, that both these suppositions are purely gratuitous, and not very probable upon physical principles; while one of them, referring to the seminal aura, rests upon an exploded hypothesis. Therefore this theory is less recommended to us by its own texture than the supposition which was made above, while it is just as gratuitous. *Secondly*, there seems no kind of reason why we should resort to any hypothesis of the kind to answer so absurd a theory as that of Grotius; because it being admitted on almost all hands that the doctrine of the Resurrection does not confine us to the very particles, at any rate to the whole particles of the earthly body, and the power of the Deity over matter being an admitted part of the system, no difficulty can be perceived in conceiving a body raised, which shall have enough of its old parts and a sufficient resemblance to the whole, for preserving personal identity through the faculties of the mind; and that personal identity is the great object in view throughout the whole inquiry.

The well known doctrine of St. Paul upon this subject is calculated to prevent the error of those who insist upon entire physical identity, and to show that there must be a change. Indeed it might seem even to justify the supposition of a much greater change than we have stated.—"That which thou sowest, thou sowest *not that body that shall be.*"—"God giveth it a body as it hath pleased him, and to every soul his own body." But this is afterwards qualified and explained. "It is sown a natural body; it is raised a spiritual body. There is a natural body and there is a spiritual body."—"The first man is of the earth, earthy the second man is the Lord from Heaven. As is the earthy such are they also that are earthy; and as is the heavenly such are they also that are heavenly. And as we have borne the image of the earthy, we shall also bear the image of the heavenly."—"Flesh and blood cannot inherit the kingdom of God; neither doth corruption inherit incorruption." (1 *Cor.* xv.)

NOTE ON THE VIS MEDICATRIX.

UNDER a former head this interesting subject was considered as connected with Evils of Imperfection. It furnishes, however, so many striking proofs of design, that some further remarks may be added.

Some have objected to the expression as grounded upon an assumption,—the hypothesis that nature acts in each instance for the purpose of remedying some mischief which has been done. But the facts are undeniable: a healing process takes place; a remedial effect is produced; and the expression only states the fact. It may be added, that the power is sometimes preventive, or prophylactic also. Thus the tendency of some poisons taken into the stomach is to induce vomiting, which throws out the offensive matter before it can produce its deleterious effects. Such, perhaps, is also the tendency of profuse perspirations, to throw off a malady in the first instance, and prevent it from taking hold of the system. When these preventives fail, the remedial power is required and comes into action.

So convinced have some anatomists been by daily observation of a kind of active power pervading and moving the system, that some speak of the vital energies as if thought as well as life could be predicated of the parts of our system. The celebrated John Hunter is an example. That great and original physiologist, being any tendency to refining, and as little certainly as any one under the dominion of vulgar prejudices, speaks familiarly of limbs and bones acting in disease, or when suffering from injuries, as if they had an intention of inflaming, and knew how to execute it. This habit of expressing himself could only have resulted from constantly observing the exact adaptation of natural operations to the uses and wants of the system in each occasion, and the exact coincidence, in point of time as well as in proportion, of the supply with the demand.

The formation of bony matter when a fracture has taken place, and the pieces of the broken bone are required to be knit together again, has been mentioned before, and the whole process is striking and instructive. First, blood is poured out into the fracture; it coagulates; soon after, very small or capillary blood-vessels shoot into the coagulated blood; the blood disappears; gelatinous matter alone remains; this gradually hardens; and bony particles are deposited which fill up the break and knit the bone. Where a dislocation has taken place there is no similar process; but as soon as the luxation is reduced, and the bones are replaced, in a very little while all the fine apparatus of the joint is restored with wonderful perfection, so as speedily to obliterate the traces of the mischief. Even where the restorative process has proved inadequate and a distortion takes place, as when by some natural defect in the firmness of some bones, they sink under the pressure of the body, a new weight being thrown upon other bones, these are strengthened additionally for the purpose of enabling them to meet the new demand upon their powers. Thus the leg and thigh bones are fortified by additional secretions of bony matter, and these are thrown up on the yielding side, and perpendicularly to the line of pressure, with as manifest a design of strengthening as is shown by those who shore or prop an old wall. Again, when after a fracture the bone of the limb is set, the ends may overlap, and thus the limb be shortened. What then shall become of the muscles which had been of a length to fit the former size of the bone? Those muscles immediately begin to shorten much beyond their original natural contraction, and they acquire a power of further contraction to suit the altered length of the bone. It is as if upon any accident happening to one part of a steam-engine, whereby it had changed its dimensions, the neighbouring parts, wholly unaffected by the accident, were of themselves to change their dimensions or their position, so that their action should also be varied, and varied exactly to suit the alteration in the part affected; thus continuing the movement of the machine, but in a different adjustment, and all without any interference of the engineer.

The throwing out of new vessels, or enlarging smaller lateral ones, in order to continue the circulation where a large or main one has been stopped up, or cut through, is another example of a kind equally striking. But the whole progress of aneurism affords perhaps the most remarkable instance of any when that progress is fully gone through. This, as is well known, is a tumour formed by the partial bursting or giving way of an artery; and if the vessel be of any considerable size, death must immediately ensue, but for a process which as immediately takes place. The blood which escapes on the rupture of the vessel coagulates and becomes solid. A kind of temporary plug is thus afforded, and time gained for a more durable repair being supplied by a more solid work being executed. Coagulable lymph is formed and thrown out, and it soon becomes firm membrane. Layer after layer of this is deposited, so that a bandage or coating is provided sufficiently strong to resist the continual pressure from the impulse of the blood. Thus the inflammatory action which ensued upon the rupture produces a new substance required for counter-acting the effects of that rupture, and enabling the artery to continue performing its functions as a conduit for carrying the blood to its destination; and this fluid itself supplies the materials with which the breach in the conduit used for carrying and distributing it is first temporarily plugged and then repaired, as if the water in a pipe were to secrete, first a sediment or lute to make the channel watertight, and then different plates of metal and braces to mend the pipes wherever its own pressure had burst them.

A similar provision is observable where a tumour has been formed in any muscular part of the body.

It results from a morbid action of those parts; but in the progress of the disease a barrier is thrown up, likewise formed out of the blood; a hard welt, of a firm condensed membrane, is formed surrounding the tumour, and interposed between it and the healthy portion of the limb.

In the case of aneurism, however, there is a still more remarkable provision added. The pressure must be relieved of the main stream of blood upon the channel, which is no longer of sufficient strength to resist it. Accordingly blood-vessels, which before had hardly been discernible, begin to work with new energy, and are enlarged in their capacity. These run parallel to the artery injured, and convey the blood so that the requisite supply continues to be afforded, but by a new system formed and in operation for the relief of the injured channel, as soon as its damage has by the first natural operation been repaired. What engineer—what Smeaton, or even Watt himself, ever constructed a pipe, such that, when it was fractured, it could not only provide itself with a plug to stay immediate mischief and enable the machine to go on, but could also provide splices for a permanent repair; and not only that, but could of itself, immediately after the accident, form new conduits and other parts exactly fitted to continue the general movement, but also to afford such relief as the injured part required, —relief exactly proportioned at once to the amount of the weakness occasioned, and to the extent of the service required? And all this without the necessity of the engineer himself being once appealed to, or any extraneous aid called in. Is there anything like this in all the works of these great men? Is there anything more marvellous even in the works of the grand Artist himself? Yes—for He too made the minds as well as the bodies of those men, and the wondrous mechanism of such minds as theirs, and those of the Newtons and La Places, which proceeded from the same hand, incomparably surpasses all the marvels of their bodily structure.

ANALYTICAL VIEW

OF THE

RESEARCHES ON FOSSIL OSTEOLOGY,

AND

THEIR APPLICATION TO NATURAL THEOLOGY.

THE great work of Cuvier stands among those rare monuments of human genius and labour, of which each department of exertion can scarcely ever furnish more than one, eminent therefore above all the other efforts made in the same kind. In the stricter sciences the "Principia" of Newton, and in later times its continuation and extension in Laplace's "Mécanique Céleste,"—in intellectual philosophy, Locke's celebrated work,—in oratory, Demosthenes,—in poetry, Homer,—* leave all com-

* If English Law were not a local learning merely, Fearne's work on Contingent Remainders would perhaps deserve to be thus ranked. In the eloquence of the pulpit, Hall comes nearer Massillon than either Cicero does, or Æschines, to Demosthenes.

petitors behind by the common consent of mankind; and Cuvier's Researches on Fossil Osteology will probably be reckoned to prefer an equal claim to distinction among the works on Comparative Anatomy. That this great performance deserves to be attentively studied there can be no doubt. But as its bulk, in seven quarto volumes, may be apt to scare many readers, there may be some use in giving a general account of the progress of the author's inquiries, and of the principal results to which they led him, and more particularly in showing their application to Natural Theology.

Long before his attention was called to the remains of animals found in various strata of the earth, in more superficial situations, in crevices of rocks, and in caves, he had, fortunately for science, been a skilful proficient in anatomy, both human and comparative. But the first steps of his inquiries concerning those fossil remains showed him how much he had yet to do before he could implicitly trust the received accounts of the animal structures. As regards the human subject, for obvious reasons, the knowledge possessed, and which the ordinary works of anatomy contain, is accurate enough and sufficiently minute. But it is far otherwise with the structure of other animals, and especially as regards their Osteology. Of this Cuvier found so many instances, that he began his investigations with examining minutely and thoroughly the bones of all those species which, or the resemblances of which, were supposed to have furnished the materials of the great deposits of fossil bones so abundant in almost every part of our globe. This, then, was the course which he invariably pursued; and he never attempted to draw any inferences respecting the fossil animal, until he had accurately ascertained the whole Osteology of the living species. There was obviously no other way of excluding mere fancy and gratuitous assumption from the inquiry, and making the science, of which he was really to lay the very foundation, one of pure reasoning from actual observation, in other words, one of strict induction.

In the course of his work there are to be found striking examples of the mistakes into which former inquirers had been led by neglecting this precaution. Partly by relying on incorrect, though generally received, descriptions,—partly by undervaluing the requisite comparisons of the fossil with the known bones,—partly, no doubt, by giving loose to fancy, observing the remains discovered with the bias of a preconceived opinion, and making the fact bend to a theory—authors had committed the most grievous errors, hastened to conclusions wholly unwarranted by the facts, and often drawn inferences which the facts themselves negatived instead of supporting. Thus M. Faujas de St. Fond, a geologist of great learning and experience, but who had upon a very scanty foundation erected a dogma, that all the fossil remains belonged to animals still found alive in different parts of the earth, and set himself to deny the novelty of all the fossil species of unknown animals, conceived that he had at length himself found among those remains two animals which, if they still existed at all, could only be found in the interior and remote parts of India. Of these supposed discoveries he published the drawings, representing two fossil heads. But Cuvier, upon examination, found one of them to be exactly the auroch or bison, and the other the common ox.* A more skilful naturalist, Daubenton, describes three sets of fossil teeth, in the King of France's cabinet, as belonging to the hippopotamus; and upon examination

* Recherches, vol. iv. p. 108.

two of these sets are found to be teeth of two new and unknown animals,* and the third alone those of the river horse; and Camper, one of the greatest anatomists of his age, fell into a similar error. Upon the discovery of some fossil bones in the Duchy of Gotha, there was a general belief that they were some *lusus naturæ*, and several medical men wrote tracts to prove it. But a nearer inspection proved them to be elephants' bones.† The town of Lucern took in earlier times for the supporters to its arms a giant, from the opinion pronounced by a very celebrated physician (Felix Plata), that the bones discovered in that canton were human and gigantic, though Blumenbach afterwards examined them, and found they belonged to the elephant. Finally, Scheutzer maintained that there were remains in different places of men who had perished in the general deluge, and supported his opinion by several instances to which he referred. Upon examination these have proved to be none of them human bones; but one set are those of a water salamander, while another belong to a newly-discovered animal still less resembling our species,

* Recherches, vol. i. p. 305.

† Ib. p. 120.

being something between a lizard and a fish.* When professional anatomists and professed naturalists could fall into such mistakes as these, there is little wonder that a statesman like Mr. Jefferson, however illustrious for higher qualities, should commit a similar blunder. He drew from the fossil bones discovered by General Washington near his seat in Virginia, and to which his attention was directed by that great man, the conclusion that they belonged to an enormous carnivorous animal, which he named the Megalonyx. Cuvier, from a more correct examination, showed the creature to have been a sloth of large dimensions, and which fed wholly upon the roots of plants.

If these examples, and they might be very greatly multiplied, evince the necessity of a cautious examination, and of a previous attention to the Osteology of animals with which we are fully acquainted, the success of Cuvier's inquiries also shows that, with due care and circumspection, the reward of the inquirer is sure. The connexion between the different parts of the animal frame is so fixed and certain, and the species run so little into one an-

* Recherches, vol. v. pp. 433 and 451.

other, that it requires but a small portion of any animal's remains to indicate its nature, and ascertain the class to which it belongs. Each small portion, so it be superficial, of bone—each little bony eminence—has its distinctive character in each species; and from one of these, or sometimes from a piece of horn, or of hoof, or a tooth, the whole animal may be determined. "If," says Cuvier, "you have but the extremity of a bone well preserved, you may by attention, consideration, and the aid of the resources which analogy furnishes to skill, determine all the rest quite as well as if you had the entire skeleton submitted to you."* Before placing entire reliance on such an induction, this great observer tried many experiments on fragments of the bones of known animals, and with a success so unvaried as gave him naturally implicit confidence in his method when he came to examine Fossil Remains.

Among those he discovered a number of animals wholly unknown, and of which no individuals have existed since the period when the authentic history of our globe and its inhabitants has been recorded. Out of the 150 which he inves

* Recherches, vol. i. p. 52. We have used the expression skeleton; the author says animal, but manifestly, from what follows, this is incorrect.

tigated about 90 were either of new orders, or of new genera, or new species of genera still living on the earth. Considered in respect to genera, there were in the 49 unknown species, 27 which belonged to unknown genera, and these genera amounted to seven. Of the remaining 22, 16 belonged to known genera or sub-genera; the total number of genera and sub-genera, to which he could reduce the whole of his fossil species, known or unknown, being 36. It must, however, be added, that it is very possible the remaining 60 also may be of new species; for as he only had the bones to examine, it does by no means follow that the living animal did not differ as much from the ones which have the same Osteology, as the mule, or the ass, or the zebra do from the horse, the jackall from the dog, or the wolf from the fox; for the skeletons of a zebra, an ass, and a horse, present the same appearance to the osteologist; so do those of the jackall, the dog, the fox, and the wolf; and yet the same bones clothed with muscle, cartilage, skin, and hair, are both to the common observer and to the naturalist animals of a different species or subdivision. This consideration is to be taken into the account as a deduction or abatement from the certainty which attends these researches; the certainty is only within certain limits; the fossil animals which now appear to resemble one another, because their Osteology is the same, may have differed widely when living; those which appear to have been of the same class with other animals that yet people the earth, may yet have been extremely different; and those which now seem to be in certain particulars different from any we or our predecessors have ever known, may differ from all that live or have lived on the earth we now inhabit, in many particulars far more striking than the varieties which their bony remains present to the osteologist's eye.*

The situations in which those remains were found, and are still to be met with in greater or less abundance, are various; but they may be reduced to three classes in one respect and to four in another: to three, if we regard only the kind of place where the bones are collected and found, in other words their mineral matrix; to four, if we regard the periods at which the earthy formations were effected, and the bones of animals living then, or immediately before, were deposited. In the former point of view, the remains are found either, *first*,

* Mr. C. once or twice adverts to this consideration; but he certainly does not bring it so prominently forward as would have been desirable.

imbedded in strata, at greater or less depth, and of various kinds, and at various inclinations;—or, *secondly*, mixed together, and with earthy matter, in caves, and in rents or fissures or breaches formed in rocks;—or, *thirdly*, scattered more sparingly, and as it were, solitarily in alluvial soil or superficial detritus, in portions of the earth, apparently while it wore its present form, and was peopled by all or most of its present inhabitants. In the latter, and the more important point of view, those remains are either found, *first*, in the beds which were deposited by the waters of a world before the existence of either human beings or the greater number of living genera of animals—as in the copper slate of Thuringia, the lias of England, the clay of Honfleur, and the chalk—in these strata the remains of reptiles are found with extinct species of marine shells, but no vertebrated animal higher than fishes;—or, *secondly*, in the strata deposited by the sea, after it had destroyed the first races, and covered the land they lived upon,—and in these beds, which at Paris lie on the chalk, are to be found only animals now extinct, and of which most of the genera and all the species differ from any we now see;—or, *thirdly*, in the strata deposited by the sea, or in fresh-water lakes,—and in these later tertiary beds are to be found animals now unknown, but resembling the present races, being different species of the same genera, or apparently of families still living, but not now inhabiting the same countries, or living under the same climates;—or, *fourthly*, in places where rivers, lakes, morasses, turf-bogs, have buried the remains of existing species; and as these changes of a limited extent have happened to the globe, constituted as it still is, those animals appear to have been for the most part identical with the animals which we still see alive in various parts of the world, at least as far as their skelelons can tell.

Paris is the centre of a most extraordinary geological district. It is a basin of twenty leagues, between fifty and sixty English miles, in diameter, extending in a very irregular form from the Oise near Compiegne on the north, to the Canal de Lory, beyond Fontainebleau on the south, and from Mantes on the Seine upon the west, to Montmirail on the east; comprehending within its circuit the towns of Paris, Versailles, Fontainebleau, Estampes, Meaux, Melun, Senlis, Nangis, and coming close to Soissons, Gisors, Beauvais, Montereau on the Yonne, Nogent on the Seine, and Condé; but not being continuous within these limits, for it is frequently cut off in islands, and everywhere towards the outline deeply indented with bays. This vast basin consists of six different formations, in part calcareous, but in some of which gypsum is so plentiful, that the quarries dug in it go by the common name of the Plaster of Paris quarries, and indeed gypsum has derived its common name from these. The lowest bed upon the chalk is composed of plastic clay, and it has covered both the plains and the caves of the district. This bed is full of fossil remains, very many of them belonging to unknown animals, and it also contains fragments of rock, which have come from a great distance. Above this bed is a layer of gritty limestone and shelly grit, of salt-water formation. Then come in succession, silicious limestone, fresh-water gypsum, and sand and grit without shells. The fourth formation is sandy, and of marine origin. The fifth has fresh-water remains and animals. The disposition of the land around and forming this Basin wears in all respects the appearance of having been broken in upon and hollowed out by a prodigious irruption of water from the south-east. Considerable corrections have since been made, especially as regards the second and third of these formations of Cuvier.

It appears that the base or bottom of the Paris Basin must have been originally covered with the sea. Different parts of the ground were then covered with fresh-water lakes, from which gypsum and marl were deposited, filled with the bones of animals that lived on their banks or in their islands, and died in the course of nature. After this deposition, the sea again occupied the ground, and deposited sand mixed with shells; and when it left the land dry for the last time, there were for a long while ponds and marshes over the greater part of the surface, which thus became covered with strata containing fresh-water shells, the base of those strata consisting of a peculiar stone found in fresh water, and occurring in many parts of France. The fossil remains in this great basin exhibit little variety of families; and the vegetable remains show that the plants were confined to palms and a few others now unknown in Europe. As the great continents, which offer a free communication throughout, are inhabited by a great variety of animals, while New Holland and the other islands in the South Seas have only a very few, and these almost all of the same family, we may conclude that the land forming the Paris basin was originally surrounded by the sea.

The deposits in the rents or fissures of the strata may now be briefly mentioned, and they present a very singular subject of contemplation. They are found all around the Mediterranean, at Gibraltar, Cette, Antibes, Nice, Pisa; in Sicily, Sardinia, and Corsica; at the extremity of the kingdom of Naples; on the coast of Dalmatia; and in the island of Cerigo. The body of the deposit is calcareous, and of the same kind in all these gaps or fissures. The same, or nearly the same, bones are everywhere found imbedded in it; they are chiefly the bones of ruminating animals; and beside those of oxen and deer, there are found those of rodents, a kind of tortoise, and two carnivorous animals. In these fissures there are many land but no sea shells; and the matter that fills them is unconnected with other strata. It follows from the first fact that they must have been consolidated before, and at the time when, the sea came over those countries and deposited shell-fish in the other strata; and from the second fact it follows that they must have been formed when the rocks, in the rents of which they are found, were already formed and dry. Hence these fissure deposits are modern compared to the strata which were formed at the bottom of the sea and of lakes. Nor does any operation now going on upon our globe bear the least resemblance, in Cuvier's judgment, to that by which those deposits must have been made. Upon this, however, great controversy has arisen among his successors.

It was necessary that we should shortly advert to the places where, for the most part, these fossil remains are found; in doing so we have anticipated a few of the conclusions deduced from the consideration of the whole subject. We are now to see what results were afforded by Cuvier's careful examination of the remains, which he instituted after he had with equal care ascertained the exact Osteology of the living animals in each case where the fossil remains appeared to offer a resemblance with existing tribes.

The *first* part of Cuvier's researches is occupied with the *pachydermatous** animals whose remains are found in alluvial deposits.

The *second* part consists of two subdivisions—in one of which are given minutely the whole details of the Paris Basin—in the other subdivision the examination of the animal remains, beginning with the

* Animals with thick skins, as the elephant, horse, hog.

pachydermatous, and then the others that accompany them, whether quadrupeds, reptiles, fishes, or birds. So that the Paris Basin is made the ground of this arrangement, and its Fossil Zoology is gone through without much regard to the general arrangement of the rest of the work.

The *third* part is occupied with the *ruminant* animals, unless in so far as one of its subdivisions, treating of the gaps or fissures of the Mediterranean, also treats of the few other animals which are there found beside the ruminant.

The *fourth* part is occupied with *carnivorous* animals—the *fifth* with *rodents*—the *sixth* with *toothless* or *edentate* animals—the *seventh* with *marine mammalia*—the *eighth* and last, and perhaps the most interesting of the whole, with *reptiles;* including the anomalous species newly discovered, which partake of the nature at once of the reptile and fish or of the reptile and bird.

As no arrangement is yet made of these fossil animals under any of the heads which we have stated, we are at liberty to adopt any order that may appear most convenient; and we shall accordingly begin with those which at first appeared to resemble the known species of the rhinoceros, the hippopotamus, and the elephant, and which a careless observer would unquestionably have confounded with these animals; but they were soon ascertained to be different.

I. Of the fossil rhinoceros four distinct species have been found;* and they are all distinguishable from the four known kinds of rhinoceros—those of India, Java, Sumatra, and the Cape. The fossil animal had a head both larger and narrower than the living kinds, and much larger in proportion to his body. He was also much lower, and a more creeping animal. He, for the most part, had either no incisive teeth or very small ones, but one species had these of a good size. One of the fossil species is distinguished from all the four known ones and from the other three fossil ones, by a still more marked peculiarity; his nostrils are divided from each other not by a gristly or cartilaginous, but by a bony partition, whence the name of *Tichorhinus*† has been given to him, the three others being termed *Leptorhinus*,‡ *Incisivus*, and *Minutus*.

The grinding teeth of the Tichorhinus, are also found to have a peculiarity which no other teeth either of any living or any fossil animal have. They

* Of these there are now nine species, five having been discovered since Cuvier's work.

† From Τυχος, a wall.

‡ From Λεπτος, slender.

are indented at the base in one of the ridges, after being worn down by use. This, as well as the bony partition, affords, therefore, the means of discovering the species. The use of the partition apparently was to support the weight of two large and heavy horns on the nose.

The history of the first of these species, the Tichorhinus, furnishes a remarkable example of the errors into which even able and expert observers may fall when they make more haste than good speed to reach a conclusion. A missionary named Campbell having sent home the head of a rhinoceros, being one of several killed close by his residence, and well known to have been so, Sir Everard Home compared it with a fossil head from Siberia, sent by the Emperor of Russia to Sir Joseph Bankes; and finding, as he thought, that it was of the same species, he very rashly inferred that the position which affirms the existence of unknown animals among the fossil remains was much weakened by this supposed discovery. Cuvier made a more accurate comparison, and found that the Cape skull was materially different from the fossil one, but resembled the head of the existing species, which Sir Everard Home had also denied. The most remarkable omission, however, of the latter was, his never looking to see if there existed a bony partition between the nostrils. This Cuvier did, and found it cartilaginous and not bony. So that the most singular of the new and unknown fossil animals belonging to this class remained still a novelty, even if Sir Everard Home had been correct in all the comparative examinations which he ever did make; and his conclusion of fact from that comparison, even if admitted to be well founded, had no bearing whatever upon the general position against which he had pointed it.

The extraordinary fact of a portion of one of these ancient and lost animal's muscular substance and skin having been found, is further to be mentioned. In a block of ice on the banks of the Wilujii, a river of Siberia, there was discovered this huge mass of flesh, about the year 1770. It was found to have longish hair upon parts on which the existing rhinoceros has only leather; consequently it must have lived in a colder climate than the present animal inhabits. But it appears to have been killed by some sudden catastrophe, and then to have been immediately frozen, else it would have undergone decomposition like the other remains of which the bones alone are left.

There are two species of living elephants, the

African and the Asiatic; the former distinguished from the latter chiefly by the length of his tusks, by a peculiar disposition of the enamel in the jaw teeth, and by never having been tamed, at least in modern times. The fossil elephant resembles the Asiatic species most, but differs in some material particulars. It has long tusks, sometimes exceeding nine feet in length; the jaw teeth are differently set; the under jaw of a different shape, as well as other bones; and from the length of the socket bones of the tusks the trunk must have been also very different. These remains* are found in great abundance both in Europe and in America, in neither of which parts of the globe are there now any living elephants of any species produced. In the same strata and caves other animals are also found both of the known and extinct classes; and occasionally shells also. The elephant's bones are chiefly discovered on plains of no considerable elevation and near the banks of rivers. They never could have been transported by the sea over the mountains of Tartary, upwards of 20,000 feet in height, which separate Siberia from the parts of Asia where the elephant now flourishes. It must be added, that, beside those bones, a still more perfect specimen of

* There are now known eight species of this fossil elephant.

the softer parts has been preserved by the action of cold than we have of the rhinoceros. In the same country, near the mouth of the river Lena, a mass of ice was found in 1799 by a fisherman, which he could not break or move; but in the course of the next summer it partially melted, when it was found to contain an entire elephant frozen. The neighbouring Tartars with their dogs, and afterwards the bears, destroyed the greater part of the flesh, but the skin and bones were saved. It was found to have hair, and even woolly hair or fur, upon different parts of the body. It must then have been calculated, like the animal of the Wilujii, for living in a climate much colder than that of India or Africa, and, like that rhinoceros, it must have been frozen immediately after its death. Its tusks were circular, and nine feet (near ten English) long.

Of the hippopotamus, two species* have been found among the fossil bones, both so different from all living animals, that every one bone of each differs from any other known bone; so that even if an error should have been committed in connecting the different bones together, there must be not only two, but more than two, new species thus discovered. These animals abound in the great deposit of fossil

* Two more species have since been found.

bones in Tuscany, in the valley of the Arno, and at Brentford, in Middlesex. There are two other fossil species, of which, however, less is known; one of these is very small, not larger than a common hog.

Three pieces of a jaw bone, with some fragments of teeth, have been found in Siberia; which upon examination prove to have belonged to a singular species, resembling both the rhinoceros and the horse, and forming probably the link between these two animals. The size is larger than the largest fossil rhinoceros. The discoverer, Mr. Fischer, has named it the *Elasmotherium*,* from the thin enamel plate which winds through the body of the tooth in a peculiar manner.

But much more is known of a lost species which approaches the elephant, although differing in some important respects both from the living and the fossil elephant. The most remarkable difference in the osteology is presented by the jaw teeth, which have the upper surface mamellated or studded with nipples; from whence Cuvier named it the *Mastodon*.† When these tubercles are worn down by use, the surface of the tooth has a uniformly

* Ελασμος, thin plate.

† Or Mastodonte, which is sometimes, but unnecessarily, rendered by Mastodonton: μαστος, mamilla.

plane or uniformly concave surface. The structure of the vertebræ shows it to have been a weaker animal than the elephant; and the belly was considerably smaller. The lower part of the fore leg was longer, and the upper joint shorter; the shoulder one-ninth shorter too. The pelvis was more depressed; the tibia and thigh bones materially thicker; and the body a good deal longer in proportion to the height. As it fed upon vegetables, and had a short neck and feet unfit for living in the water, it must have had a trunk; and it also had tusks. It seems to have fed upon the softer parts of vegetables, and to have inhabited marshy ground. Six species* have been discovered of this animal, chiefly differing from each other in the teeth; and of these six, two only are well known. The mastodon was long supposed to be peculiar to America, and was sometimes called the Ohio animal; but there have since been found teeth in different parts of Europe, evidently belonging to the two better known species; and the other four kinds are, to all appearance, European.

In the same strata with the remains of elephants, rhinoceroses, and other animals both of extinct genera and species, are almost everywhere found the bones

* Five more species have since been discovered.

and teeth of horses, very nearly resembling those of the animal now so well and universally known. It yet happens that for want of due attention to a branch of anatomy more familiar to us than any except the human, naturalists have constantly fallen into error in examining fossil bones. Thus Lang, in his history of the figured stones of Switzerland, took a horse's tooth for a hippopotamus's; and Aldrovandinus in one work describes teeth of that class as giants', and in another as horses'; while several authors have confessed that they could not tell to what tribe such remains had belonged. Cuvier did not, therefore, deem himself released from the duty of fully examining the common horse's osteology, merely because of the frequent and minute descriptions which had previously been given of it; and his intimate acquaintance thereby obtained with the nature of every bone and tooth, has enabled him to pronounce with confidence upon the existence of horses like our own among the unknown animals which inhabited the earth before the vast revolutions that changed both its surface and its inhabitants. He has, however, justly noted the fact that there is no distinguishing the bones of the horse, the ass, the mule, and the quagga; so that very possibly these remains may have belonged to any of those animals;

and very possibly also to none of them, but to some fifth species, now, with the mastodon and other contemporary animals, extinct. The same remark is of course applicable to the bones of the hog and the wild boar, found occasionally among other fossil remains.

The tapir family in many important particulars resembles the rhinoceros; and those are often found in the same tertiary strata with the rhinoceros, elephant, and mastodon, several species now wholly extinct, but allied to the tapir. Two of these must have been of prodigious size, the largest 18 feet (19¼ English) long and 11 (nearly 12 English) high.* But there are other species, to the number of twelve at least, whose size differs little from that of the tapir; the bones are somewhat different however, and particularly the teeth, which, from the eminences or ridges upon them, Cuvier made the ground of the genus, to which he gave the name of *Lophiodon*.† It is in different parts of France that all these species were first found; the smaller ones always in strata of fresh-water shells, and in company with remains of either unknown land animals, or crocodiles and other river

* This is now better known, and is called the *Dinotherium*.

† Λοφιον, a small hill, eminence, or ridge.

animals now found in hot climates; and in several places the strata in which they occur, have been covered over, after they had been deposited and their bed consolidated, with strata of an origin unquestionably marine. By far the greater part of fossil remains, both those which have been already described and those which we are afterwards to consider, have been found in sandy, or calcareous, or other earthy strata. But some few are also found in imperfect coal or lignite. In the part of the Appenines where that range meets the Alps there is a tertiary coal stratum, and in it have been found two new genera of pachydermatous animals, and a third in the fresh-water deposit near Agen. Cuvier calls these *Anthracotheria.**

The general conclusion which is to be derived from the important branch of the inquiry of which we have been analyzing the resulting propositions, is partly zoological and partly appertains to geology. The former portion of it is, that more than thirty kinds of land animals have left their fossil remains in the strata now forming dry land, but deposited under water; that of these, seventeen or eighteen†

* Ανθραξ, coal. Of these seven species are now known.

† According as the Elasmotherium is allowed to be sufficiently distinguished or not.

are now extinct, and have been wholly unknown since the earth was peopled with its present inhabitants, six or seven being of a genus now unknown, the others being new species of known genera; that twelve or thirteen kinds have, as far as their bones are concerned, the appearance of having belonged to the species which still inhabit the globe, although their identity is far from certain, depending only upon the similarity of their skeletons; and that animals of genera now almost confined to the torrid zone used formerly to inhabit high and middling latitudes. The geological portion of the conclusion is that some of these fossil remains have been buried by the last or one of the last revolutions to which our planet has been subjected, as they are in loose and superficial strata, whilst other remains in the tertiary strata appear generally to have come from deaths in the course of nature, though some of these too must have perished by a sudden revolution.

II. The Paris Basin presents, in great abundance, the remains of herbivorous pachydermatous animals of two distinct genera, each comprehending several species, and all alike unknown in the living world. The animals to which some of them approach the nearest are the tapirs; but they differ even generically from these, and from every other known tribe. The inquiry into which Cuvier entered for the purpose of ascertaining to which set of bones each particular piece belonged, so that he might be able to restore the entire skeletons by putting together all the parts of each, was long, painful, and difficult in the highest degree. He had first to connect the two bones of the hinder feet together, in each instance, by minutely examining the relation of the pieces to one another; and this process could only be conducted by deriving light from the analogies of other and known animals. He then had the different bones of the fore feet in like manner to put together, in order to restore those fore feet. Next the hinder and fore feet of each animal were to be connected together. Afterwards he had to mount upwards and connect the bones of the body with the several feet. The teeth and head must next be referred to the limbs. Then the vertebræ and then the trunks were to be restored; and then other bones, not yet accounted for, were to have their places found. The result of this most elaborate and perplexing investigation, the details of which occupy the fifth part of a large quarto volume, and are illustrated by between sixty and seventy admirable plates, containing between six hundred and seven hundred figures of bones, fragments of bones, and congeries of bones, may be stated shortly thus:—There are of the first genus, which he denominates *Palæotherium*,* six, or perhaps seven, species,† principally distinguished by the teeth and the size, as far as the bones are concerned, but which, probably, were much more widely different when alive. One of these resembled a tapir, but was only a foot and a half in length, being about the size of a roebuck. Another was nearly three feet high, and the size of a hog. A third was between four and five feet in height, and about the size of the horse or the Java rhinoceros. It had feet thicker than a horse's, and a larger head; its eyes were very small, its head long, and it had a snout protruding much over its under jaw and lip. In a specimen of one of these species, the first now mentioned, there were actually found some of the animal's softer parts, certain flexible filaments, which, upon being burnt, gave an animal smell, and were manifestly portions of the nerves or blood-vessels. Besides these three species, three, and possibly four others, were distinguished, one the size of a hare.

The other genus was termed by Cuvier *Anoplo-*

* Παλαιος, ancient; θηριον, wild beast.

† Eleven species are now known.

therium,* and of these, two species, at the least, are distinguishable.† The first, or *common* anoplotherium, is about the size of an ass, being four or five feet high, and its body four feet long, but with a tail of three feet long; it was probably an animal that lived partly in the water, as it appears made for swimming like an otter. But it has a peculiarity of structure which is to be found in no other animal whatever; its feet are cloven, but have two separate and distinct metacarpal and metatarsal bones, which are soldered together in other animals; it has also its teeth contiguous, while all other animals except man have them apart. The other species, or *secondary* anoplotherium, resembles the former, but is only the size of a common hog. But beside these anoplotheria properly so called, four other cognate species are found, one of the size and appearance of a gazelle, one the size of a hare, and two of the size of a guinea pig. A curious specimen gives the very form of the anoplotherium's brain, a cast of it remaining in the earthy mass. Its size is extremely small, and Cuvier infers from this that the animal was exceedingly stupid.

* Ανοπλος, unarmed, without tusks.

† Six species are now ascertained.

All these animals are found in the Paris Basin; but bones of the palæotherium have been discovered elsewhere, namely, at Orleans, Aix in Provence, Montpelier, and Isell. As the specimens from those other places were extremely rare in Cuvier's time, he could not have the same certainty respecting them as from the more copious collections obtained in the Paris district. But he could distinguish at least three different species.

Beside these two new genera, the palæotherium and anoplotherium, the Paris Basin affords two other new genera of pachydermata, the one, called *Chæropotamus,** resembling animals of the hog kind —the other, *adapis*, very small, being about a third larger than the hedgehog, which it also resembled in structure. There are found, too, the remains of five or six kinds of carnivorous animals, one of them being of enormous size, and resembling a tiger. Another has projecting bones to support a bag or purse as in the kangaroo kind; but it is of a genus of marsupial animals now found only in America, being a sort of opossum. The Basin, besides, affords a considerable number of tortoise remains, some fish bones, and even perfectly complete skeletons of fish, and ten species, at least, of birds, all

* There are now three species known.

now unknown, but one of which resembles the Egyptian ibis. It is very remarkable that in one specimen, brought to Cuvier while his work was printing, the windpipe was preserved, and the mark or mould of the brain appeared upon the surface of the gypsum.

III. Of ruminating animals the fossil deposits present many remains. There are of the deer, beside divers that closely resemble known species, no less than twelve* species wholly unknown among the existing inhabitants of our earth. One has enormous horns, six feet from tip to tip, and of this animal we know nothing among existing species, though it comes nearest the elk. Two kinds are somewhat like roebucks, and of that size. The fissures of the Mediterranean give six new species,† of which that found at Nice is like an antelope or a sheep.‡

* No less than twenty-eight species are now known.

† In the Résumé to Parts III. and IV., Cuvier says, "Of the six deer found in alluvial deposits, one with large horns is entirely unknown; of the four in fissures, three are unknown, at least in any but most distant countries. Another, that of Orleans, is quite unknown, as are the two species of lagomys found in the fissures."

‡ A thirteenth new species was at one time supposed to have been found in the Swedish province of Scania; but Cuvier, before the last volume of his work was printed, had reason to believe that this animal belonged to one of the tribes formerly known, and still living in the north of Europe.

None of our common oxen are found in a fossil state, unless in morasses or peat bogs, where they have certainly been buried while the globe's surface was in its present condition, and peopled as we now find it. But animals of the same genus certainly existed in the age of the elephant and rhinoceros, and of the extinct species.* There prevails no small uncertainty as to the identity of the fossil bison and musk buffalo with the living species of the former in Europe and of the latter in America; but the remains which have been found of a kind of ox, appear different from any known species, and it appears that no buffalo resembling either that of the East Indies or that of the Cape, has been found in any place.

The conclusions, both zoological and geological, from this part of the investigation and from the examination of the remains found in the Paris Basin, in every respect tally with those to which we were led by a consideration of the pachydermatous remains under the first head of the inquiry.

IV. There are found in caverns both in France, Germany, Yorkshire, and Devonshire, and in the fresh-water formation of Val d'Arno, in Tuscany,

* Of these there are now seven ascertained.

the remains of many animals, some extinct and others no longer inhabitants of the same temperate latitudes, but confined to the frozen and the torrid zones. By far the greater part of these animals belong to the carnivorous class, except in the Yorkshire caves, where many of the herbivorous kind are also to be found. In the foreign caves the bear is the most numerous, and presents extinct species. In the Yorkshire caves (at Kirkdale) the hyæna predominates. In the German caves hyænas are comparatively few, and in Val d'Arno not more numerous. In Kirkdale there are very few bears. The race of lions and tigers is much more rare than any of the others. Not above fifteen have been found in Germany, while there have been found hundreds of bears; and in Yorkshire, where hyænas abound, very few lions and tigers are traceable. Of the wolf and fox, some are found, but not so many in Yorkshire. There is also a very large kind of dog traced, which must have been five feet in height and eight in length from the mouth to the tail.

Of bears it appears, after a very close examination, that there are found, at least, two species* larger than those now known, and a third which, both in size and other particulars, so nearly approaches the

* Seven mo:e have since been added.

common bear, that Cuvier does not regard it as a new species. But it seems as if the one found in Tuscany formed a third kind of animal now extinct.

The hyæna* is found not only in the caverns and other quarries where the bear abounds, but also in the alluvial strata with the elephants and rhinoceroses. In Kirkdale cave his dung has been distinctly recognised by a comparison with that of living hyænas; and the particular crack which he makes in the bones of the beasts devoured by him to get at the marrow, has, in like manner, been identified by actual comparison. Nevertheless the fossil animal differs from the living one in some material respects, particularly in size, and in having his extremities both thicker and shorter. The caverns contain two species† of a huge animal of the felis (or cat) kind, considerably larger than the lion or the tiger, beside some few resembling living species in size. One is between one-eighth and one-ninth larger than the lion, and has its trunk more convex in the lower outline. A new, but smaller, species of the felis kind is also found in the Mediterranean fissures.

In the dog tribe there has been found a wolf or dog,‡ but more probably the former, which differs

* Now eight species. † Now fifteen.

‡ Ten species are now known.

though slightly, from any known species, in having the muzzle shorter in proportion to the skull; and also a species has been observed clearly new of the same genus. We as yet only know of it by two of his jaw teeth, found at Avaray, near Beaugency. He must have been eight feet long and five high. The Paris Basin affords, likewise, another new species of the dog kind, but not materially varying in point of stature. The common fox, however, is found, and also the dog and wolf, in the caves.

The caves afford a considerable number of bones of the weasel and glutton,* closely resembling the existing species. The latter animal is only known now in the higher latitudes; but in the caves we find his remains mixed with those of animals belonging to the temperate and the torrid zones.

It is thus shown by the inquiries which comprise the third and fourth part of this great work, that the former inhabitants of these regions were wholly different from the present population. Even the animals of hot climates here found, and referable to existing genera, must have differed entirely from those species which survive in the torrid zone, because they could exist in a temperature now wholly foreign to their nature. The rein-deer and the lion, the sloth and

* Of the fossil gulo two species are now ascertained.

the elephant, all found in the same places, show that the climate of those latitudes remains nearly the same, but that their inhabitants have been changed.

In all these researches one blank is immediately perceptible. There are not only no human remains whatever, but there are none of apes or of any of the genus of quadrumanes. Animals far less in size, and whose bones would much more easily have perished, as rats and mice, have left their skeletons with those of the largest beasts; but of the monkey tribe no vestige whatever is to be discovered; and the conclusion is inevitable, that the strata were deposited, the fissures filled, the caverns strewed with bones, at an age anterior to the existence of that tribe, as well as to the creation of our own species. Thus it was when Cuvier wrote.*

V. Beside the animals of the *Rodent* description, found in the Paris Basin and the Mediterranean fissures, rabbits, lagomys, field mice, there are several others in the alluvial strata and caverns,—some apparently of known, and others, certainly, of unknown kinds. The hare has been traced at Kirkdale; the beaver near the Rhine; two new

* This refers of course to the state of discovery in Cuvier's time. There are remains of the monkey said to have been lately discovered in the South of France and in the Himalaya Mountains; it is said also at Calcutta. But the proofs are not clear.

species* of the beaver near Rostoff, in the south of Russia; another species, also unknown, at Œningen.

VI. The toothless or *Edentate* animals afford some varieties still greater than those to which our attention has as yet been directed. None of the known species of this tribe are to be found in any of the strata, fissures, or caves in Europe. But three genera entirely new, with two of which at least there are ample materials for becoming acquainted, have been found in America, and these are deserving of our best attention.

The first is the animal named by Jefferson, from the size of his feet, or rather what he supposed claws, the *Megalonyx*,† and respecting which he fell into an error as we formerly stated. Cuvier preceded his examination of this as of all other animal remains by a thorough investigation of the osteology of living animals of this family; and it is the result of his careful inquiry that the bones found in America and described by Jefferson, and of which both casts and drawings were sent over, as well as a tooth, belonged to an animal of the sloth tribe, but wholly new and now quite extinct. The tooth was cylindrical, and worn down on the

* Now four are known, and three of lagomys.

† Two species are now known.

top, but cased round with enamel like a sloth's, and not at all like a cat's. In the paw, the second phalangal bone was symmetrical. This bone is curved and not symmetrical in animals that raise up and draw back the claw, as all the cat kind do. The first phalangal bone, too, was the shortest; whereas the lion and others of the cat kind have that bone the longest. But from the known species of sloth it differs most strikingly in its stature, which was equal to that of the largest oxen, those of Hungary and Switzerland, and a sixth larger than the common kind.

The second of these new animals has been termed *Megatherium,* from his great size, and the remains are found in South America. From his teeth it appears that he lived on vegetables, but the structure of his very long fore paws and nails shows that it was chiefly on the roots. He possessed also good means of defence, and so was not swift of foot. His covering seems to have been a thick and bony coat of mail like the armadillo's. His length was twelve feet and a half (near thirteen feet and a half English), and his height seven feet (about seven feet and a half). From the sloth he differs not only in size but in other particulars; for example, his fore legs are much nearer the length of his hinder legs than in the sloth, which has the former double the latter. But, on the other hand, the thickness of the thigh bone in the megatherium is much greater than in any of the known sloth tribe, or indeed any other animal either known or extinct; for the thigh bone is about half as thick as it is long.

The third of these new animals was known to Cuvier only by one fragment which he examined. It was a toe; and from a careful discussion of its form and size he inferred that the animal belonged to the edentate tribe of Pangolins, and that, if so, its length must have been twenty-four feet (twenty-six English), and its height in the same enormous proportion. The bones were found in the Palatinate near Eppelsheim.*

VII. The course of our analysis has now brought us to the family of the *Sea Mammalia*, and these supply new food for wonder. So different from the bones of any living animals are those remains which have been examined, that a new genus is formed consisting of several species, and bearing the same relation to the cetacea, or animals of the whale tribe, that the mastodon, palæotherium, and anoplotherium do to the

* Subsequent discoveries have made it probable that this toe belonged to the Dinotherium.

pachydermata, or that the megalonyx and megatherium do to the edentata. He terms the genus *Ziphius* from its having a sword-like head. One of these was found near the mouths of the Rhone. The dimensions are not given by Cuvier, but from the drawing the head appears to have been about three feet in length. The remains of a second species of ziphius were found thirty feet under ground at Antwerp, and between nine and ten under the level of the sea at low water. The head is considerably larger than that of the first mentioned species. The head of a third species is found in the museum at Paris, but with no account of its history.

Beside this new genus, there are other cetacea of new species discovered among the fossil bones. At Angers a Lamantin of an extinct species has been traced. The remains of a dolphin, which must have been twelve or thirteen feet long, and different from all the known species, have been found in Lombardy. In the Landes another dolphin, which must have been nine or ten feet in length, has been discovered. A third kind of dolphin, different from any now living, has been found in the department of L'Orne, while a fourth, also found in the Landes, nearly if not wholly resembles the ordinary dolphin. In Provence a cetaceous animal of an unknown species is found, somewhat like the hyperodons.

In the neighbourhood of the Ochill hills in Scotland the fragments of a whale's bones have been found in a recent alluvial stratum, at only eighteen inches depth, with a part of a deer's horns near. It must have been a whale of some size, as the vertebræ were eighteen inches broad, and one of the ribs ten feet long. But it is most probably one of a kind still existing in our seas, from the place where it was found.

In the mountains near Piacenza there have been found the bones of a small whale. Its length was twenty-one feet (near twenty-three of ours) and its head was six feet (near six feet and a half) long. The place where these bones lay was a clay stratum with numberless shells all round, and oysters clinging to the bones. This animal was in a tertiary formation, six hundred feet above the plain of Italy. It appears to be of a new species.

In the very heart of the city of Paris have been found the bones of another whale, far larger, and of a species wholly unknown. Its head must have been fifteen or sixteen feet long, and its body fifty-four or fifty-five. It was found in a compact sandy bed in digging under the cellar of a wine-merchant.

The conclusion to which these Researches unavoidably lead is that the earth in its former state did not differ more widely in the races which inhabited it than the sea did—that ocean which was itself the great agent in producing many of the changes that have at various times swept away one race of living creatures from the surface of the globe, and mixed up their remains with those of animals engendered in its own bosom.

VIII. We have now reached the last and the most singular portion of these Researches; the examination of the *Reptiles* whose relics are found in many of the stratified rocks of high antiquity.

In the calcareous schist, near Monheim, whence the stones used in lithography are gotten, a new species in the crocodile family is found, whose length must have been about three feet. At Boll, in Wirtemberg, another, apparently of the same kind, has been discovered. At Caen oolite quarries, a different and equally unknown species is traced; its body is between four and five feet long, and its whole length thirteen. Others of this family have been found in the Jura, and there they are accompanied by the fresh-water tortoise. At Honfleur another species is found, and the remains of two other unknown kinds have been discovered near Harfleur and Havre.

Beside the remains of crocodile animals found in these more ancient strata, there are many also found in the more recent beds, where the bones of the palæotheria and lophiodons are deposited. The Paris Basin, the marl pits of Argenton, Brentford, and other places have furnished these specimens. But whether they were of different species from those new ones found at Monheim, Caen, and Honfleur, the examination which they had undergone in Cuvier's time was too imperfect to determine. They have since been shown to be different.

It deserves to be remarked of the new species of crocodiles, that their difference from the known kinds exceeds in manifest distinctness that of almost any other animals which are of the same genus, and do not differ in size; for the vertebræ instead of being, as they are in the crocodiles now alive, concave in the front and convex behind, are convex in front and concave behind. This at once furnishes a very triumphant answer to those doubts which have been raised as to the novelty of the species, and still more signally discomfits the speculations of those who fancy that the difference perceived in fossil bones have been caused by change of temperature or of diet, or by the passing from the living to the petrified state.

The examination of fresh-water tortoises, of the genus trionix, whose remains are found in the plaster quarries and other strata offers similar results. Thus at Aix in Provence a trionix of a new species is found. Another species, also new, is found in the Gironde; and two others have been traced less distinctly in the gravel beds of Hautevigne (Lot et Garonne) and of Castelnaudary.*

Fossil fresh-water tortoises, of the genus emys, give the same results. They are found in the molasse of Switzerland, in the Sheppy clay near London, and in the limestone ridges of the Jura.

Fossil sea tortoises offer the like appearances. One of an unknown species is found near Maestricht, the genus being still living in the sea, and familiar to our observation. So that altogether the examination of tortoise remains leads to the same inferences of islands having existed in the ocean at

* Eight species have now been traced.

a former period, inhabited chiefly by reptiles or oviparous quadrupeds, and before the creation of any considerable number of the viviparous orders.

As we proceed towards the close of these Researches the subject rises rather than falls off in curiosity and interest. We now come to the family of lizards, by which is here understood all the old genus of Lacerta (Lin.), excepting the crocodile and salamander tribes.

In the celebrated fossil fish deposits of Thuringia are found the remains of a monitor, of a species somewhat varying from the known species in two particulars, a greater elevation of the vertebral apophyses, and a longer leg in proportion to the thigh and foot. Remains of a similar aspect occur in France near Autun, and in Connecticut in North America.

In the strata of fine and granular chalk near Maestricht, between 400 and 500 feet in thickness, are found the remains of a huge reptile, which Mr. Faujas represented as a crocodile, following the opinions of the people in that neighbourhood; but so celebrated an anatomist as Adrian Camper was not to be thus deceived, and he proved it to be an animal of a new genus, related to the monitor, and also to the iguana; it seems to be placed between the fishes on the one hand and the monitors and iguanas on the other. But the size constitutes its most remarkable difference when compared with these. They have heads five or six inches long; his was four or five feet, and his body fifty. He was therefore a lizard exceeding the size of a crocodile; just as the extinct tapir was the size of an elephant, and the megalonyx was a sloth the size of a rhinoceros. It appears that, like the crocodile, he was aquatic and could swim; and that his tail was used as a scull, moving laterally in the water, and not up and down like the cetacea, an order to which the elder Camper at first rashly referred him.

In the canton of Meulenthal, at Monheim, ten feet below the surface, and near some kinds of crocodile remains, bones were discovered of another unknown sub-genus of the order Saurus, and which Cuvier calls Geosaurus, and places between the crocodile and the monitor. It was apparently twelve or thirteen feet long, that of Maestricht being fifty.

A large animal of this family is found to have been an inhabitant of the same ancient world. At Stonesfield, in the neighbourhood of Oxford, Dr. Buckland discovered his remains in a bed of oolitic calcareous schistus under a solid rock of forty fee thick. The thigh bone is two feet eight inches in length, which would seem to indicate a body in the whole forty-five feet long. But even if his tail were not in the proportion of the lizard's, as this calculation assumes, his length must be, according to the crocodile's proportions, thirty feet. This animal approaches the geosaurus of Monheim, and also, in other respects, has some affinity with the crocodile and monitor; but in size he greatly exceeds the crocodile and comes nearer the whale. His voracity must, from his teeth and jaws, have been extreme. He was also an amphibious animal; for his remains are surrounded with marine productions. The genus has been called *Megalo-saurus.* Teeth and bones of the same genus have been since discovered in Tilgate Forest, Sussex. Mr. Mantel has found in the same place the thigh bone of a much larger animal. Other reptiles have been found in the Muschelkalk quarries near Luneville.

But there are animals of the family of saurus yet more strange, if not for their size, at least for their anomalous structure and habits. A reptile is found of a genus so extraordinary as to comprehend within itself the distinguishing nature both of the lizard and the bird. It has a very long neck, and the beak of a bird. It has not, however, like a bird, wings without fingers to strengthen them; nor has it wings in which the thumb alone is free like a bat; but the wings spread by a single long finger, while the other fingers are short, and with nails like the fingers of ordinary apterous (or unwinged) animals. From these circumstances Cuvier has named this genus* the *Pterodactylus.*† It was first discovered by the late Mr. Collini, a Florentine, settled at Manheim, and formerly attached to the family of Voltaire, of whom he published some memoirs. The skeleton, nearly perfect, was found in the marly stone beds of Aichstadt in the county of Pappenheim; but Mr. Collini fell into very great mistakes respecting the genus of the animal, which he supposed to be of marine origin, from not accurately investigating its osteology. The celebrated Sœmmering contended that it was one of the mammalia, resembling a bat, and other naturalists held the same opinion. But Cuvier has most satisfactorily shown, chiefly from its jaws and vertebræ, its shoulder blade and sternum, that it is between a bird and a reptile, a flying reptile. The tail is

* There are now ten species observed.

† Πτερον, wing; δακτυλος, finger.

extremely short, and this indicates the animal to have used its wings chiefly for locomotion; indeed, from its very long neck, it must have had great difficulty in either walking or crawling. When at rest, it must have stood like a bird on its hind legs, and also, like some birds, have bent back its long neck in order to support its very large and heavy head. Another species of the same genus, having a much shorter beak (for that of the former is longer than the whole body), has also been found near the same spot. It is much smaller. Very scanty remains of a third species also occur, found in the same quarries. Its size must have been nearly four times greater than that of the kind first mentioned, and it must have presented one of the most monstrous appearances which can be conceived, according to our present experience of animal nature.

The two last discoveries among the animals of a former world, which these Researches have disclosed, remain to be mentioned; and they are, in the eyes of the naturalist, the most wonderful of the whole, although to an unlettered observer they may appear less strange than the tribe we have just been surveying. One of them has the muzzle of a dolphin, the teeth of a crocodile, the head and breast of a lizard, the fins or paddles of a whale, but four instead of two, and the back or vertebræ of a fish. This has been named the *Ichthyosaurus*. The other, being apparently nearer to the lizard, has been called the *Plesiosaurus*;* and has also four paddles, like those of a whale; the head of a lizard, and a long neck like that of a serpent. Both are found in the older secondary strata of the globe; in the limestone marl or greyish lias, filled with pyrites and ammonites, and in the oolite beds of the formation called Jurassick. They are both chiefly found in England, and were first discovered there.

Sir E. Home, in 1814, made the first step in the discovery of the Ichthyosaurus; having obtained some bones found on the Dorsetshire coast, thirty or forty feet above the level of the sea. He gradually obtained more of these remains, until 1819-20, when the discovery was completed. But he seems to have been unfixed and variable in his opinion respecting the animal; and after believing for some time that it was partly a fish, he ended by believing it to be no such thing, and changed its name from ichthyosaurus, which Mr. König had given it, as early as 1814, to *Proteosaurus*, supposing it to

* Πλησιος, near.

have some affinity with the proteus as well as the lizard.

The ichthyosaurus is most abundant in the lias strata in the lower region of the Jura formation. Its remains are not confined to Dorsetshire; they are found in Oxfordshire, Somersetshire, Warwickshire, and Yorkshire. But at Lyme they abound as much as those of the palæotherium do in the pits of Montmartre at Paris. Some few specimens are found near Honfleur and at Altorf; in Wirtemberg, also, a nearly complete skeleton has been discovered. Four* distinct species were ascertained by Cuvier, chiefly differing from one another by their teeth, that is to say, as far as their osteology goes.† In the general features of their bones they all approximate to one another. The head resembles that of the lizard, although with material differences and even having some other bones. The eyes are

* Four species have since been added to these.

† It cannot be too steadily kept in mind that when a specific difference has once been ascertained, so as to distinguish one of these extinct races from another, the amount of that difference is no measure at all of the diversity which may have existed between the two animals. Tribes the most unlike have general resemblances in the bones, the substratum on which the muscular parts are placed. Witness the ease with which unlearned persons, nay, even naturalists carelessly observing, have taken the skeletons of lizards for those of men.

extremely large, differing in this from all the greater animals both sea and land. The cavity in some specimens is above a foot in diameter. Each eye is protected by a shield of bone, composed of several pieces knitted together. The vertebræ are very numerous. In some specimens as many as ninety-five are to be seen; and these differ entirely from the vertebral system of the lizard, resembling rather that of fishes, for they are flat like backgammon, and concave on both sides. The animal has four fins, or paddles, each composed of six rows of small bones, nearly one hundred in all, and so fitting into one another, that he could paddle about by means of them, moving with more elasticity than if the bones had formed a single piece. The teeth are sharp. This creature could only breathe the air, and so must often have come up to the surface. Yet, again, he could only move in the water, and was still less able to crawl on land than even the sea-calf. The length, in some cases, reaches to twenty-four or twenty-five feet. In the strata where these bones are found there are many of the cornu ammonis and other marine shells, and remains of crocodiles exist in the same strata.

The plesiosaurus was first observed in 1821, by

Mr. Conybeare and Mr. Delabeche; and in Cuvier's time its remains had only been found in England, unless those discovered at Honfleur belong to this genus. The discovery was fully made in 1824. The distinguishing feature, the long neck, has many more vertebræ than even a swan's. In the fine specimen from Lyme there are in all eighty-seven vertebræ, of which thirty-five belong to the neck and twenty-five to the tail. The vertebræ, though their axis is very short, resemble the crocodile's more than the lizard's. The teeth are pointed and slender. The paddles consist of many bones, in rows like those of the ichthyosaurus; but they taper more, consist of fewer pieces, not above fifty, and are longer than those of the ichthyosaurus, nor do they form a kind of pavement like his. Five species* of this animal were distinguished by Cuvier. That found at Lyme appears to have been seven or eight feet long; but other species, from one jaw bone which has been discovered, must have reached the length of twenty-eight feet.

The eighth and last part of these Researches which we have just surveyed, is remarkable, as regards the

* Three have since been added.

skill and diligence of the illustrious author, for two particulars. *First,* the extraordinary success of his indefatigable investigation from very scanty materials derives especial attention. In some cases he had only one or two bones to examine and to reason from. In others he had a far greater number; sometimes he had the whole skeleton in scattered parts; in a few instances the whole together in their natural juxtaposition and connexion. But he found where he had many bones, that from a single one, or from two, he could have reached the very same conclusions which the examination of the whole led him to. This was observable in a very remarkable manner when he investigated the mosasaurus, or saurus found at Maestricht. He had not examined more than the jawbone and the teeth when he knew the whole animal; but he says that a single tooth discovered it to him: he had got the key; after that every other part fell in at once of itself into its proper place. *Secondly.* Although he was not the discoverer of either the ichthyosaurus or plesiosaurus, and had to tread on ground which his eminent and able predecessors had gone over, his researches even here were quite original. He collected all the evidence, whether by drawings, descriptions, or models, of what had been before them; but he also enlarged his collection of facts by numberless specimens both of the same kind which they had examined and of different kinds never submitted to their view. He investigated the whole as if the field had been still untrodden and the soil yet virgin; and accordingly his work, even in this subordinate branch is far from being a repetition; his inquiries far from being a mere reiteration of theirs. Where he does not vary or extend the results at which they had arrived, he carefully confirms their propositions, and ascertains the truth of their learned conjectures; so that he adds to the precious monuments of his predecessors, by either enlarging the superstructure or strengthening the foundation.

That such a guide to our inquiries is worthy of all confidence, no one can doubt. That even his authority, the weight of his opinion, is very great would be a proposition as indisputably true, if in matters of science it were lawful for the learned to pay any deference to mere authority; yet even here ignorant men may bow to him, and receive his doctrine with a respect which they might be justified in withholding from others. But his system makes no such appeal, and requires not to be received upon terms like these. He has given us without any reserve every particular which his whole researches presented to his own view, and preferring the risk of being tediously minute to the chance of leaving any point unexplained, or any position without its needful proof, there is not a fragment of bone which he has ever examined, and on which he raises any portion of his philosophy, that he has not both described with the fulness of anatomical demonstration, and offered to the eye of his reader in the transcript of accurate and luminous engraving. His work is accompanied with between forty and fifty maps and sections of strata, above 250 plates representing upwards of 3,800 skeletons, bones, teeth, and fragments. These are all presented to the examination of the expert, in their connexion with the author's description both of what the diagrams can, and of what they cannot, fully represent. But they are also presented to the uninformed, who can, by attentively considering them, institute a comparison between the structure of known and living animals, and those of which the earth's strata contain only the remains. Giving Cuvier only credit for having correctly written down what he observed, and accurately represented in his figures the subjects of his examination, we are enabled to see the whole ground of his reasoning; we can mark the points in which a fossil animal resembles a living one, and those in which the two differ; and we have even a higher degree of evidence in behalf of the author's conclusions than we have in reading Sir Isaac Newton's experiments upon light, because every thing in this case depends upon configuration, which a drawing can accurately represent, whereas much in the optical case must needs turn upon appearances observed by the experimenter, and which no drawing can convey to our apprehension.

If again we compare the certainty and fulness of the proof in this case with that which we have in examining any anatomical proposition, or any doctrine of natural history, whether of animals or of plants, we shall still find it of a separate and higher kind. For in those branches of science much more is necessarily left to description. The question here is always one purely osteological as regards the animals; and osteology is of all branches of anatomy, whether human or comparative, the one where most depends upon mere figure, and where of consequence the reader can approach most nearly to the observer in weighing the proofs on which his demonstration rests. The geological matter bears but a small proportion to the zoological in these inquiries. It is indeed of the highest importance; but it is incapable of much doubt, and admits of no mistake or imposition—for the strata where the different animal remains have been found are well known, and, in the very great majority of cases, are of easy access to all. The sciences of geology and mineralogy are sufficiently certain, at least for the main purposes of the inquiry; the names and description of the beds of the globe's surface are the portions of those sciences upon which no doubt or difficulty can exist; and the great body of Cuvier's results, remains unaffected by any differences of opinion upon speculative geology.

Thus the comparison stands as to the degree in which the evidence is made plain to the reader of Cuvier's researches, and the reader of other records of discovery in the inductive sciences. But let us extend our view a little further, and compare the proofs before us in these volumes with those reasonings upon which the assent of mankind has been given, and is continued unhesitatingly, to the great truths of the mixed mathematical sciences.

The reader of the "*Principia*," if he be a tolerably good mathematician,* can follow the whole chain of demonstration by which the universality of gravitation is deduced from the fact that it is a power acting inversely as the square of the distance to the centre of attraction. Satisfying himself of the laws which regulate the motion of bodies in trajectories around given centres, he can convince himself of the sublime truths unfolded in that immortal work, and must yield his assent to this position, that the moon is deflected from the tangent of her orbit round the earth by the same force by which the satellites of Jupiter are deflected from the tangent of theirs, the very same force which makes a stone unsupported fall to the ground. The reader of the "*Mécanique Céleste*," if he be a still more learned mathematician, and versed in the modern improvements of the calculus which Newton discovered, can follow the chain of demonstration by which the wonderful provision made for the stability of the universe is deduced from the fact that

* It is the object of the Analytical View of that great work in this volume to make the demonstration, the proof on which the Newtonian system rests, so easy as to be followed by persons little skilled in mathematical science; but the remarks in the text will, it is to be feared, always remain well founded. The like may still more be said of the Analysis of La Place's Mécanique Céleste.

the direction of all the planetary motions is the same, the excentricity of their orbits small, and the angle formed by the plane of their ecliptic acute. Satisfying himself of the laws which regulate the mutual actions of those bodies, he can convince himself of a truth yet more sublime than Newton's discovery though flowing from it, and must yield his assent to the marvellous position that all the irregularities occasioned in the system of the universe, by the mutual attraction of its members, are periodical, and subject to an eternal law which prevents them from ever exceeding a stated amount, and secures through all time the balanced structure of a universe composed of bodies, whose mighty bulk and prodigious swiftness of motion mock the utmost efforts of the human imagination. All these truths are to the skilful mathematician as thoroughly known, and their evidence is as clear as the simplest proposition in arithmetic is to common understandings. But how few are there who thus know and comprehend them? Of all the millions that thoroughly believe those truths, certainly not a thousand individuals are capable of following even any considerable portion of the demonstrations upon which they rest, and probably not a hundred now living have ever gone through the whole steps of those demonstrations. How different is the case of the propositions discussed by Cuvier and his predecessors! How much more accessible are the proofs on which their doctrines repose! How vastly more easy is a thorough acquaintance with the "*Recherches*" than with the "*Principia*" and the "*Mécanique Céleste!*" How much more numerous are they who have as good reason for fully believing the propositions, because as great facility of thoroughly examining the proofs, as first rate mathematicians can have for assenting to Newton's third book, and Laplace's great theorem, or as common readers have for admitting any of the most simple truths in the easiest of the sciences!

The extraordinary truths unfolded by the "*Recherches*" we have had an opportunity of stating in detail. But it is necessary to revert to some of the more general conclusions in their more immediate connexion with the great subject of these volumes. The Illustration derived to theological inquiry from the powers of inductive investigation in this branch of science, and the Analogy found between the two kinds of demonstration, was stated in the Introductory Discourse; but these form by no means the whole contribution which this new branch of knowledge furnishes to Natural Religion. Before the nature and extent of that aid could be understood, it was necessary that the details of the science itself should be considered, and its general principles unfolded, together with the grounds upon which they rest. We are now more particularly to make the application.

To the geologist, as Cuvier has well observed, the vast periods of time over which the phenomena that form the subject matter of his inquiries have extended, offer the same kind of obstruction as the astronomer finds from the immense space over which his researches stretch. The distance of time is to the one as great a difficulty as that of space is to the other in prosecuting his researches. Yet as the properties of light, and its relation to media artificial or natural, furnish a help to the senses of the astronomer, so the endurable nature of the principal portions that compose the framework of animal bodies give invaluable assistance to the labours of the geologist and anatomist, supplying records which it is as physically impossible he should have in any history of past changes on the globe, as it is that the naked eye of the astronomical observer should penetrate into boundless space. The most minute bones of small animals, even their cartilaginous parts, and the most delicate shells of sea or river fishes, are found in perfect preservation. These shells are found, too, on ground now and for ages lying high above the level of any waters, in the middle of the hardest rocks, reaching the summits of lofty mountains, lying in vast layers of a regular form and solid consistency, and which seem to demonstrate the proposition that the sea in former ages was spread over the regions where those strata were formed, and lay there long and quietly. The level parts of the earth, which to an observer who only regards its surface seems always to have been in its present state, can hardly be penetrated in any place without showing that it has undergone such revolutions and been under the sea for ages; while the bottom of the ocean has at those remote periods been dry land. But when we ascend to greater heights, we find the same proofs of former changes; marine remains often show themselves on Alpine summits, but their kinds vary much from those of the lower regions; they are exposed to view by the layers in which they lie imbedded being no longer horizontal and buried deep under ground, but nearly vertical, broken in pieces, and thrown variously about. These strata have for the most part been of a formation long prior to that of the horizontal ones, and were at one time displaced, and elevated and rolled about; the ocean was the great agent in their formation as in that of the strata which it afterwards deposited horizontally around them; the ocean, too, was the agent which, after having first deposited, afterwards dislocated and raised them into rocks, promontories, and islands, amidst which the strata still found horizontal were laid.

This ocean, at different times, not only held in solution different dead matter, but was inhabited by animals of kinds that exist no more. When it last left the earth and retreated into its present position, the only one in which we have ever known it by actual observation, its inhabitants nearly resembled those which still live and swarm in its waters. But at more remote periods, and when forming its more ancient deposits, it was the receptacle of animals of which not a living trace now remains; animals all whose species are extinct; animals of genera absolutely different from any now known, and which sometimes united together in one individual frame, parts now only found separate in distant and unconnected tribes.

Again, the intermixture of land animals and of fish the inhabitants of fresh water only, with those of marine origin, shows that several successive irruptions of the ocean must have taken place, and that after it remained covering the land during successive periods, it retreated successively, and left that portion of the globe dry. Nor can there be any doubt that large portions of the earth now uncovered and inhabited by the human species and other tribes of living animals had, before it was last covered by the sea, been dry, and been inhabited by a race of animals of which their fossil remains are all that we can now trace.

It is probable, too, that many of these mighty revolutions have been sudden, and not effected by gradual incroachments upon the earth, to destroy its inhabitants. The examination of masses of flesh belonging to some of the race destroyed by the last change, and preserved by the frozen water in which they were imbedded, seems to prove that the death of the animals, and their envelopement in water, the coagulation of the water, and the introduction of a frozen climate, were simultaneous; for the putrefactive process had not commenced till thousands of years after the destruction of life, when, the ice being thawed, the exposure to heat and air began the decomposition. But the sudden violence by which these last changes were effected is equally conspicuous in the transport of huge blocks from one part of the country to another in which they were manifestly strangers.

But we ascend to greater heights on the surface of the globe, and we find the scene changed. We are now upon the vast and lofty chains of solid rock which traverse the central parts of the different continents, separate the rivers that water and drain them, veil their summits in the clouds, and are capped with never melting snows. These are the primitive mountains; formed before any of the other new made strata whereof we have already spoken, because they penetrate them vertically; and even these primeval rocks show by their chrystallization and occasionally by their stratified forms that they, too, were once in a liquid state, and deposited by waters which anciently held them in solution and covered the places they now fill. In these, as we ascend to the most ancient, no animal remains at all are found.

The shells and other marine productions so abundant below, and in the more recent layers of the globe, here cease altogether to exist. The primeval rocks, therefore, were first held in a liquid state, and afterwards deposited, by an ocean which contained in its bosom no living thing; an ocean which before covered, or washed, a continent, or islands, on which life never had existed.

There is also little doubt, according to Cuvier, though we give not this as an incontestable proposition, that the prodigious changes which we have been contemplating must have been operated by a force wholly different from any that we now perceive in action upon any portion of the globe. The power employed to work some of the displacements of which we see the traces is shown remarkably in the insulated masses, found removed from great distances, and lying still at vast heights. On the Jura, at near 4,000 feet above the level of the sea, are found blocks of granite evidently carried from the Alps, one of which, containing 50,000 cubic feet of stone, has been removed and placed in its present position after the formation of the strata on or among which it lies,—strata, the materials of which do not fill its interstices, but have been rent and broken by its fall. None of the operations now observed on the earth's surface satisfactorily explain either this or the other revolutions in question. The effects of weather, either in the fall of rain, or in alternate freezing or thawing of water, though sufficiently powerful and very beneficial upon a small scale in decomposing stones and pulverizing earths, are confined within comparatively narrow limits. The action of rivers in wearing down their banks, and changing the position of their beds, is restricted to those banks and beds, and is of slow and almost imperceptible operation, unless in some cases of rare occurrence, where a mountainous eminence being gradually undermined may fall and dam up a river and cause a lake to be formed, or where a lake may be let out of its reservoir by the wearing away of some ridge forming its dam or head, and so inundate the country below—events barely possible be it observed, and of which the period of authentic history records scarcely any instance. Then the incroachments of the sea are even more gradual than those of rivers; nor can any proof be found, in all the time over which authentic human annals reach, of a material change in position of the ocean with respect to its shores; the utmost it has ever done being to wear away an isthmus here and there,* or cover a mile or two of low and flat coast.† The wonderful force of a column of compressed water, in a vertical fissure connected with a subterraneous sheet of it, however shallow, but filling a broad space—the resistless power of such a column to move about any superincumbent weight—has, perhaps, been too little taken into account as an agent in effecting changes on the earth's surface. But these operations must be all merely local. Volcanic action is still more topical in its sphere; and though violent enough within these narrow limits, produces consequences wholly confined to them, and unlike those which are under consideration. Lastly, whatever effect could be produced by the motion of the earth is of incomparably a more slow and gradual kind than any now enumerated. The motion of the poles round the plane of the ecliptic, and the nutation of the axis,

* There seems reason, from some ancient authorities, to believe that the Isle of Wight was once a peninsula when the tide was out, to which tin, the staple of the ancient British exportation, was carried in waggons at low water to be shipped for Gaul.

† The estate of Earl Godwin in Kent, now covered by the sea, is one of the principal examples of this kind of change; and there must clearly be great exaggeration in the accounts given of it.

are movements of this kind, and never exceed certain narrow limits. The rotation of the earth has a regular and defined tendency to accumulate matter towards the equator, and flatten our globe at the two poles, but no other; and certainly neither a sudden nor a violent effect can be operated by this means.

The result of the Researches upon the fossil bones of land animals has demonstrated those changes still more incontestably than the examination of the remains which have been left by the inhabitants of the ocean; both because, as they must have lived on dry land, their being found in strata deposited by water proves that water has covered parts of the continent formerly dry, and also because their species being fewer in number and better known, we can now certainly tell whether or not the fossil animal is the same with any still living on the globe. Now, of the one hundred and fifty quadrupeds examined by Cuvier, and whose remains are found deposited in different strata of our continent, more than ninety are at present wholly unknown in any part of the world; nearly sixty of these are of genera wholly unknown, the rest being new species of existing genera; only eleven or twelve are so like the present races as to leave no doubt of their identity, or rather of their osteology being the same; while the remaining fifty, though resembling in most respects the existing tribes, as far as the skeletons are concerned, may very possibly be found, on more close survey, and on examining more specimens, to differ materially even in their bones. Nor is it at all unlikely that, of the whole one hundred and fifty, every one would be found to be of a race now extinct, if we could see their softer parts as well as their bones and their teeth. But the relation which these different species of ancient animals bear to the different strata is still more remarkable and more instructive in every point of view.

In the *first* place, it appears that oviparous quadrupeds, as crocodiles and lizards, are found in earlier strata than those containing viviparous ones, as elephants and others. The earth which they inhabited must, therefore, have existed and been watered by rivers before the chalk formation, because they are found under the chalk in what is termed the Jurassic formation.—But, *secondly*, among the strata subsequent to the chalk formation, the unknown genera of animals, palæotheria, anoplotheria, are only found in the series of beds immediately over the chalk. A very few species of known genera of viviparous quadrupeds are found with them, and also some fresh-water fishes.—*Thirdly*. Certain extinct species of known genera, as elephants, rhinoceros, are not found with those more ancient animals of extinct genera. They are chiefly found in alluvial earth, and in the most recent tertiary strata, and all that we find with these extinct species are either unknown, or of more than doubtful identity with any now existing. Again, those remains which appear identical with the known species are found in recent alluvial earths, and places which seem to belong to the present world.—*Fourthly*. We have seen that the most ancient secondary strata contain reptiles and no other quadrupeds. None of the rocks at all contain any human remains; nor were any remains of the monkey tribe, or any of the family of quadrumanes found in Cuvier's time, if indeed they are observable even now. In turf-bogs, in rents and cavities, under ruins as well as in cemeteries, human skeletons are from time to time found; but not a vestige of them or of any human bone in any of the regular strata, or of the fissure deposits, or of the caves and caverns which abound with all the other animal remains. Whatever human bones have been found, were undoubtedly placed there by human agency in recent times.

For Cuvier has examined with the utmost care all the instances which were pretended to afford proofs of human remains. He closely investigated several thousands of the bones in the Paris basin, and in the deposits of Provence, Nice, and others. All which had ever been supposed to be human he found to be either animal bones, or bones of men accidentally placed among the others, or in some other manner satisfactorily accounted for. The skeleton supposed by Scheutzer to be a man's, and which he made the subject of his book, "*Homo Diluvii Testis,*" a century ago, has been already adverted to. Cuvier undertook the complete examination of it. The first skeleton which formed the subject of Scheutzer's argument was found near Amiens. Thirty years afterwards another was discovered; but its possessor, Gesner himself, raised grave suspicions that it was some lower animal's remains. A more complete one than either was afterwards found. Cuvier has engraved this, together with Scheutzer's copied from his own book—and how any person could, upon the bare inspection, ever have conceived that either was a human skeleton is truly incomprehensible. But Cuvier has further engraved a land salamander, whose osteology he had, after his admirable manner, thoroughly examined, and its likeness to the fossil remains shows it to be of the same genus, though of a wholly new species, above six times larger. He enters at large into the details of the difference between these remains and the human skeleton. But a further demonstration of their nature was reserved for him when, in 1811, at Leyden, he had access to the actual fossil itself of Scheutzer, and was permitted to remove a portion of the incrusting stone. He did this with the salamander by him, and predicted the kind of bones that would be discovered by the operation. The success of the experiment was complete; and to show the difference between this skeleton and a human subject, Cuvier had the satisfaction of also discovering a double row of small and sharp teeth, studding the fringe or border of the large circular mouth. In 1818, he had an opportunity of repeating this examination upon the last found specimen, which is now in the British Museum, and with exactly the same result. It is therefore demonstrated, as clearly as any fact in the whole compass of physical science, that these bones belong to a race wholly different from the human species, and indeed from any species now existing on the face of the globe. Finally, places where human bones have for many centuries been deposited with the remains of animals, as the ground under ancient fields of battle, have been examined, and it is found that the one are quite as well preserved as the other, and have not suffered more decay. The importance of establishing the conclusion that no human remains are to be found in the strata of the earth will presently appear, and is the reason why we have dwelt upon the evidence in some detail.

If we next inquire at what period the last great change took place, although of course no records can remain to fix it, yet we have some data on which to determine the limits of the question. The progress of attrition in the larger rivers, as the Dnieper and the Nile, and also the formation of downs where they approach from the sea, has been observed, as on the coast of the Atlantic in the south of France; and the results indicate no very remote antiquity as the age of the present terraqueous distribution; certainly not more than 5,000 or 6,000 years. Of these, history only goes back about 3,000. Homer lived but 2,800 years ago. Genesis cannot have been written earlier than 3,300 years back. Even the earliest Chinese monuments that are authentic reach but 2,255 years. The astronomical remains of the East, when closely examined, especially the Zodiac, prove nothing of that extreme antiquity which was at one time ascribed to them. Nor do the mines, such as those of Elba, from which similar inferences were formerly deduced, show, since their more accurate examination, any thing of the kind. Indeed none of the conclusions they lead to can be regarded as at all of a certain kind. The general result of the Inquiry, then, is, that at a period not more remote than 5,000 or 6,000 years ago, a mighty convulsion covered with the ocean all those parts of the globe then inhabited by man and the other animals his contemporaries, and left dry those other portions of the earth which we now inhabit. The few remains of the races then destroyed have served to people this new world; it is only since this period began that we have entered upon the progressive state of improvement in which our race has advanced; and to this period whatever historical monuments we possess of the globe or its inhabitants are confined. But it is equally clear that this inhabited earth, then left dry for the last time, had previously undergone several revolutions, and had been alternately dry land and covered with the ocean, more than once, or even twice, before this last revolution. We have access more particularly to examine the condition and population of the earth when it was last inhabited, that is, when the sea left it the last time but one. We are now living in the fourth æra or succession of inhabitants upon this earth. The first was that of reptiles; the second that of palæotheria; the third of mammoths and megatheria; and it is only in this present or fourth æra in succession that we find our own species and the animals which have always been our companions.

We are entitled then to affirm that, with respect to animal life, three propositions are proved, all of great curiosity, and still more, when taken either separately or together, all leading to conclusions of the highest importance—

First—that there were no animals of any kind in the ocean which deposited the primary strata, nor any on the continent which that ocean had left dry upon its retreat;

Secondly—that the present race of animals did not exist in the earlier successive stages and revolutions through which the globe has passed;

Thirdly—that our own species did not exist in those earlier stages either.

Now the conclusion to which these propositions leads, and which indeed follows from any one of them taken singly, but still more remarkably from the whole, and most especially from the last, is that a creative power must have interposed to alter the order of things in those early times. That an interposition of this kind took place, the last and most important, about 6,000 years ago, is highly probable from the physical and natural evidence alone which is before us, and to which alone in this work reference can be made. But the date is not material. If at an uncertain period before the present condition of the earth and of its inhabitants, there were neither men nor the present race of creatures, wild and domestic, which people the globe, then it follows that between that period, whensoever it was, and the earliest to which the history of the world reaches back, an interposition of power took place to create those animals, and man among the rest. The atheistical argument, that the present state of things may have lasted for ever, is therefore now at an end. It can no longer be affirmed that all the living tribes have gone on from eternity continuing their species; and that while one generation of these passed away and another came up in endless and uninterrupted succession, the earth abided for ever. An interruption and a beginning of that succession has been proved. The earth has been shown not to have for ever abode in its present state; and its inhabitants are demonstrated, by the incontrovertible evidence of facts, to have at one time had no existence. Scepticism therefore can now only be allowed as to the time and manner of the creative interposition; and on these the facts shed no light whatever. But that an act of creation was performed at one precise time is demonstrated as clearly as any proposition in natural philosophy, and demonstrated by the same evidence, the induction of facts, upon which all the other branches of natural philosophy rest.

It is wholly in vain to argue that the sea or the earth, or the animals formerly existing and now extinct, or any other created beings, or any of the powers of nature, as we know it, or as it has ever been known, could have made the change. It is difficult enough to conceive how these known forces could ever have destroyed the earth's former inhabitants. But suppose the approach of some comet or other body at different times produced the vast tides by which the land was successively swept, this will not account for new species and new genera of living creatures having sprung up both to inhabit the land and to people the waters. An act of creation—that which would now be admitted as a direct interposition of a superior intelligence and power—must have taken place. This is the sublime conclusion to which these Researches lead, conducted according to the most rigorous rules of inductive philosophy, precluding all possibility of cavil, accessible to every one who will give himself the trouble of examining the steps of the reasoning upon which they repose, and removing doubt from the mind in proportion as their apprehension removes ignorance. It is an invaluable addition to the science of Natural Theology, and forms a chapter as new in kind as any of the new animal species are in Natural History.

Such are the benefits conferred upon the great and fundamental argument of Divine Intelligence and contrivance by the recent discoveries in Fossil Osteology. The evidence of design in the combination and mutual adaptation of the parts of extinct animals, we pass over as only a multiplication of proofs sufficiently numerous before. But the other branch of Natural Theology, that which investigates the Divine Benevolence, also derives aid from this new quarter. We now refer to the argument maintained in the Dissertation upon the Origin of Evil, and also to the theories which were there very respectfully considered, and diffidently and reluctantly found to be unsatisfactory. The late interesting discoveries have thrown new light upon both these subjects of discussion, and the authors of some of the systems which we examined may appeal to the improved state of our knowledge respecting the Chain of Being, as we certainly do make our appeal to it upon what appears to be a more solid ground of argumentation.

The doctrine respecting the Chain of Being is admitted to be incomplete as regards the matter of fact, inasmuch as we find many and large blanks in the series of animated creatures known upon our globe. Whatever other objections, therefore, were competent against this theory, an additional one was, that little appearance of a Chain of Being seems discernible in the universe. Now, the supporters of this doctrine have certainly a right to maintain that the blanks are filled up in a very remarkable manner by the recent discoveries. For the new species of animals discovered to have existed in former states of the globe, unquestionably fill up some of the most remarkable chasms in our series of living animals. Thus the chief blank was always observed in the pachydermatous animals, the fewest in number, the least approaching one another, and the whole tribe the most removed from others. Now most of the new and extinct kinds of quadrupeds belong to this class, and we have had occasion to observe how links are supplied between race and race hitherto appearing altogether distinct.

But although we may not be justified in reposing great confidence in the argument drawn from the plan of a Chain of Being as applied to the subject of positive evil, there is another point of view in which the subject may, with perfect safety, be considered. As far as regards mere defect, mere imperfection, it is most important to consider whether the plan of Divine Providence may not have been to create a succession of beings rising one above another in attributes; say merely of intelligent beings thus differing in their approaches to perfection. The importance of this consideration cannot fail to strike the observer when he reflects that there is no possibility of separating one of the greatest of all positive evils, death itself, from mere defect or imperfection, as was observed in the Dissertation already referred to; not to mention many other kinds of evils arising from mere imperfection,—as all that proceed from weakness, from ignorance, from defect of mental energy, as well as mental perspicacity. All these evils, and all their various consequences, originate in mere defect or imperfection. Therefore it is of no little moment in this important argument that we should be able to derive any new light to guide our steps upon that part of the ground which belongs to defect or imperfection.

Now the late discoveries certainly afford us some such lights. They show as plainly as the evidence of facts can show anything, that there was a time when this globe existed with animals to people it, but without any beings at all of the human kind. The sounder opinion certainly is, that there have been a succession of stages through which the earth has passed, with different races of animals belonging to each period; that in the earliest age of all no animal life existed; that this was succeeded by another in which reptiles were found to flourish, and that subsequent periods were marked by other successive races of animated beings. But as this is the subject of controversy, we shall only say that there have been two eras, one in which inferior animals only existed without man, and the other in which we now live, and in which our species are the principal inhabitants of the globe. This is admitted by all who have considered the evidence; and they who the most strenuously deny the other doctrines of Fossil Osteology avow their implicit belief in the great proposition, that the relics of an age are clearly discovered in which man had no existence.

Now this position is most important with a view to our present argument. It appears that there was a time when the Creator had not brought into existence any being above the rank of the lower animals. It follows that the divine wisdom had not then thought fit to create any animal endowed with the intelligence and capacity and other mental qualities of the human species. If an observer had been placed in that world, and been called upon to reason regarding it, what would have been his reflections on the imperfections of animated nature? Yet, after a lapse of some ages, those defects are all supplied, and a more accomplished animal is called into existence. The faculties of that animal, and his destinies, his endowments and his deficiencies, his enjoyments and his sufferings, are now the subjects of the observer's contemplation and of his reasoning. What ground has he now for affirming that a more perfect creature may not hereafter be brought into existence—a creature more highly endowed and suffering far less from the evils of imperfection under which our race now suffers so much? No one can tell but that as many of the former inhabitants of the globe are now extinct—tribes which existed before the human race was created — so this human race itself may hereafter be, like them, only known by its fossil remains; and other tribes found upon other continents, tribes as far excelling ours in power and in wisdom as we excel the mastodon and the megatherium of the ancient world.

It is to be further observed, that no uncreated being can, by the nature of the thing, have any right to complain of not being brought into existence earlier. The human race cannot complain of having come so late into the world; nor can any of the tribes created before us complain that they were less perfect than a species, the human, which did not then exist. Have *we*, then, the inhabitants of the present world, any better reason to complain that the new, as yet unknown, possible creatures of a future period of the universe have not as yet come into existence? It must be confessed that the extraordinary fact, now made clearly and indisputably* known to us, of a world having existed in which there were abundance of inferior creatures, and none of our own race, gives us every ground for believing it possible that Divine Providence may hereafter supply our place on the globe with another race of beings as far superior to ourselves as we are to them which have gone before us. But how inconceivably does this consideration strengthen and extend the supposition broached in the Dissertation upon Evil! How strikingly does it prescribe to us a wise and wholesome distrust of the conclusions towards which human impatience is so prone to rush in the

* The kind of controversy which may be raised, but never has been raised on this point, is discussed in the next dissertation.

darkness of human ignorance! How loudly does it call upon us to follow the old homely maxim. "When you are in the dark, and feel uncertain which way to move, stand still!" How forcibly does it teach us that much—nay, that all which now we see as in a glass darkly, and therefore in distorted form and of discoloured hue, may, when viewed in the broad and clear light of day, fall into full proportion and shine in harmonious tints!*

* Dr. Paley, in his twenty-fifth chapter, assumes, that whenever a new country has been discovered, with new plants and animals, these are always found in company with plants and animals which are already known, and possessing the same general qualities. From hence he derives an argument for the unity of the First Cause. Mr. Dugald Stewart also infers from the supposed identity of animal instincts in all ages, that the laws of physical nature must have always been the same, otherwise these animals could not have continued to exist.

Now, *first*, as to Dr. Paley's assumption. It certainly appears too large, even as regards the existing species and the present state of the globe; for there seem to be some places where all the animals are peculiar. But be that as it may, the fact assumed is by no means necessary for the support of Dr. Paley's conclusion in favour of the Divine Unity. It is extremely probable that in some former stages of our globe there were no animals whatever of the same tribes with those which to us are familiarly known. Yet can there be any doubt that in their structure the same degree of skill is observable as far as their only remains enable us to judge, and can we hesitate to believe, that were there other parts before us, we should in those find as much artist-like contrivance as in the existing races of animals? Indeed we may go further and assert, that there is every ground for supposing that the same kind, as well as an equal measure of skill, is to be traced in the lost as in the existing tribes, and that, consequently, the characteristic argument will equally apply here. The proof of this in the structure of the alimentary canal, which Cuvier was not acquainted with, will presently be considered.

Secondly. With respect to the observation upon instinct, unquestionably some doubt may be raised by the new discoveries; for we cannot feel any confidence in the assertion that the animals, whose skeletons alone remain, were endowed with instincts similar to those now in being, more especially the tribes of anomalous description, such as the pterodactylus and ichthyosaurus. We have never seen in life any animals combining the various forms which seem to have met in these extraordinary creatures. We cannot, therefore, feel entire confidence in the belief that their habits or instincts resembled those of any combination of animals so dissimilar,—still less can we comprehend a harmonious union of the instincts proper to birds with those peculiar to reptiles, which yet the pterodactyli seem formed to obey. Dark, however, as is this department of the subject, we have abundant ground, from the preponderating weight of analogy, for resting satisfied that all their instincts, whatever they may have been, were nicely adjusted to their bodily powers, and that both their bodies and their instincts were as nicely adapted to the laws of matter and of motion.

It would be improper not to mention at the close of this Analytical View, that the science of Palæontology was much indebted to some able and learned men who were contemporaries of Cuvier. The examination of the Paris Basin, as regards its mineral character, was almost wholly the work of Brongnart. and it is allowed to be a model in that kind. Cuvier's brother, also, ably assisted him in the botanical department. The labours of Lamarck in conchology are so universally known as to need no further mention; and among other names may be stated that of Miller of Bristol, as having made valuable contributions to these inquiries.

LABOURS OF CUVIER'S SUCCESSORS.

Many learned men were attracted by the discoveries of Cuvier, and devoted themselves to the cultivation of the same science. During the last twelve or fifteen years of his life they had joined in similar pursuits, and many of his opinions were modified, and many of his researches were materially aided, by their diligent and successful inquiries. As far as regards the general connexion between Organic Remains and Geology, indeed, another inquirer had appeared in the field as early as himself, the laborious, modest, and sagacious William Smith, a civil engineer, who, unassisted and almost unknown, had been prosecuting his researches into the mineral state of England, and performed certainly the most extraordinary work that any single and private individual ever accomplished—the delineation of the strata of the whole country, in a set of underground maps, which he published in 1815, and followed afterwards with a work upon the relation between these strata and their Organic Remains. Although the results of his investigations were published thus late, he had many years before communicated the greater part of them freely to his private friends. It must be confessed that few men of greater merit, or more unassuming, have ever adorned any walk of science, and few have ever made a more important step in assisting the progress of discovery.

The other able persons who have cultivated this branch of science are certainly endowed with greater learning, that is, book learning, than Mr. Smith could boast of, beside attending closely to actual observation in the field. Some of them, too, may fairly claim a high place as men of profound and original views. Where so many excel and prefer claims so undeniable to the gratitude of the world, it is invidious as well as difficult to make a selection, the rather as, happily, we still have the great benefit of their continued assistance. In Italy, Brocchi; in Switzerland, Studer, Hugi, Charpentier, and Agassiz, the able and zealous disciple to whom Cuvier gave up the department of fossil ichthyology, when composing his work on Comparative Anatomy; in Germany, Von Buch, Kaup, Count Münster, Goldfuss, Rosenmuller, Wagner, and the justly celebrated Humboldt; in Russia, Fischer; in Belgium, Burtin, Omalius, Dumont; in France, Beaumont, Brongnart, Blainville, Prevost, Bouè, Brochant, Geoffroy; and in England, Conybeare, Mantell, Lyell as incident to his Geological Treatise, Clift, Delabeche, König, Hibbert, Broderip, Fitton, Bakewell, Greenough, Owen, Murchison, Professor Sedgewick, and Dr. Buckland. These, it is believed, are all, except Brocchi, fortunately still alive, and still actively engaged in the same interesting inquiries, though some of them rather confine their study to the geological portion of the subject. If from the brilliant assemblage the names of Sedgewick and Buckland were selected, but, as regarding Fossil Osteology, the latter especially, private friendship could hardly be charged with officiously assuming to be the organ of the general voice—but, indeed, to record such merit might well seem presumptuous, where the panegyric is far less likely to reach after times than the subject of its praise.

The labours of Cuvier's successors, as far as regards his doctrines, belong to one or other of three classes: to the progress which they have made in examining the fossil remains of former worlds, or conditions of our globe;* to the arguments which they have advanced in opposition to or in support of his theory respecting the relation

* The notes to the Analysis of Cuvier contain statements of the numbers of new species discovered since his time.

that subsists between those animal remains and the strata in which they are found; and to the arguments adduced for or against his opinions respecting the formation and age of those strata. It may be proper to mention the things done under each of these heads, although the last is of comparatively little importance to the purpose of the present work, and the second is of considerably less moment, as regards Cuvier's proper subject, than the first.

I. Among the extinct mammalia of the pachydermatous order, we mentioned one which Cuvier referred to the tapir genus, but pronounced to have been of a gigantic size. He only had seen the jaw teeth of the animal. But since his time other important parts have been found, chiefly at Epplesheim, in Hesse Darmstadt; and a genus *Dinotherium* (having four species) has been established, of which this species is termed *giganteum*, his length having been apparently not less than eighteen or nineteen feet. His distinguishing peculiarity is the having two enormous tusks, which are bent downwards like those of the walrus, but are placed at the front end of the lower jaw, so as to bend below the chin. Dr. Buckland has shown by most cogent arguments that he must have lived chiefly in the water, and these tusks in all probability were used in supporting him, anchored as it were, to the side of the river or lake while his huge body floated, as well as employed in digging for the roots upon which his teeth show that he fed.

Notwithstanding somewhat scanty materials, Cuvier had described and, as it were, restored the megatherium with extraordinary skill. But a further importation of bones from South America has enabled observers in this country to throw some additional light upon the structure and habits of this singular animal. These bones were found in the bed of the river Salados in Buenos Ayres, a succession of very dry seasons having brought the water unusually low. Mr. Clift, of the Surgeon's Museum, a most learned and skilful comparative anatomist, and pupil and assistant of John Hunter, examined them fully, and found many very singular particulars not before known respecting this animal. Among other things it appears to have a bony partition between its nostrils (septum narium) like the rhinoceros tichorhinus. The structure of its teeth indicates that they are formed by perpetual growth like the elephant's tusks, and not like his teeth by renewal. The enormous size of the tail never could have been conjectured from the analogy of the elephant and other pachydermatous animals. It was composed of vertebræ, of which the one at the root had a diameter of seven inches, and the diameter from the extremities of the processes was no less than twenty-one inches. If then allowance be made for the muscle and integuments, it could not have been less than two feet in diameter at the root, and six feet in girth. There can be little doubt that it was used both as a weapon of defence and to support the animal in conjunction with part of his large feet, while the others were employed in digging or scraping away the earth in quest of his food. The fore feet were a yard long, and the bones of the fore legs were so constructed that the limb could have a lateral or rotatory horizontal movement for the purpose of shovelling away the soil. The bone of the heel is also of extraordinary length. The proportion of his bones to those of the elephant is very remarkable. The first caudal vertebra in the megatherium being twenty or twenty-one inches, in the elephant it is barely seven. The circumference of the thigh in the former is two feet two inches, in the latter one foot. The expanse of the os illii in the former no less than five feet one inch, in the latter three feet eight inches. The bony cover of the hide has also been now more fully examined. It was about an inch in thickness, and so hard as to resist all external violence. The cumbrous movements of this unwieldy creature exposing it to many kinds of danger, the hide served to defend it from some enemies, and the weight and strength of its limbs and tail enabled it to destroy others; escape from any by flight being quite impossible. Mr. Clift informs me that he has found in the region of the pelvis small lumps of adipocire. So that we have here an additional instance of the softer parts of an extinct animal still preserved in a state to which flesh is now often reduced by decomposition in water.

Mr. Darwin (grandson of the celebrated physician and poet) has found in South America many interesting remains. Among these are the bones of an edentate, between the megatherium and armadillo (largest kind); those of a huge rodent in size equal to the hippopotamus; and those of an ungulate quadruped the size of a camel, and forming the link between that class and the pachydermata.

In the lias stratum of Lyme Regis there was found in 1828, by Miss Anning (to whose skill in drawing, as well as her geological knowledge, Cuvier often acknowledges his obligations), a new species of pterodactylus with very long claws, and hence Dr. Buckland gave it the name of *Pter. Macronyx*. It appears to have been the size of a raven.

In 1824, Mr. Mantell discovered in the Tilgate sandstone, in Sussex, the remains of an herbivorous reptile allied to the iguana genus, but vastly larger; and he gave it the name of *Iguanodon*.* Other parts of the animal have since been found in different places, as in Purbeck, and in the Isle of Wight. Mr. Murchison found a thigh bone three feet seven inches long; and in 1829, a metacarpal bone, of six inches long by five wide, was found in the iron sand, and a vertebra as large as an elephant's. The opinion of Cuvier referred the large thigh bone clearly to Mr. Mantell's reptile, whose dimensions must therefore have been enormous, though it was not carnivorous.

In 1834, a large proportion of the skeleton was found in the Rag quarries, near Maidstone. This confirmed all the previous conjectures as to the bones separately discovered. The length of this monstrous reptile is calculated to have been seventy feet from the snout to the tip of the tail, the tail to have been fifty-two feet long, and the body fourteen feet round.† Mr. Mantell also discovered

* This discovery had been made before the last edition of Cuvier's book, and is mentioned, though shortly, in the Analysis.

† Geol. Trans. N. S. vol. iii. pt. 2.

in 1832, in Tilgate Forest, the remains of a lizard, which may have been twenty-five feet long, and was distinguished by a set of long, pointed, flat bones on its back, some rising from it as high as seventeen inches in length. He called it *Hylæosaurus,* from being found in the Weald.

There were found in 1836, a great collection of fossil bones in the department of Gers, in France, in a tertiary fresh-water formation. Above thirty species, all mammalia, were traced, and of these the greater part were new extinct animals, but all were of extinct kinds; two species of the dinotherium; five of the mastodon; a new animal allied to the rhinoceros, and another to the anthracotherium; a new edentate; and a new genus between the dog and racoon; but the most singular and new of the whole is the under jaw of an ape, which appears to have been thirty inches in height. But we must be very cautious in giving our assent to this, until we are better informed of the position where the jaw was found. It is certainly possible; but after the history of the Gaudaloupe skeleton, clearly human, as clearly found among fossil remains, but now universally admitted to have been a recent deposit, we may pause before concluding that a deposit contrary to all other observations of fossil bones should have occurred in any tertiary formation.*

In the time of Cuvier, at least before the completion of his great work, our knowledge was so scanty of the fossil osteology of the East, that we doubt if any allusion to it is ever made by him. Three most important contributions to this branch of science have since extended our knowledge in that direction, and a rich addition may soon be expected from Mr. Clift's labours upon a large recent arrival.

The first was by my excellent friend Mr. Craufurd, who, travelling in the Burman empire, was fortunate enough to discover a great number of fossil remains near the river Irawadi. These he generously gave to the Geological Society, and Mr. Clift proceeded to examine them with his wonted assiduity and skill. Among them were traced two new species of mastodon, in addition to the M. gigas, and M. angustidens, of Cuvier. One is termed by Mr. Clift *Latidens,* from the breadth of his jaw teeth; and the bones of his face exceed in size those of the largest Indian elephant. The other

* I have lately seen an appearance of a stratum of calcareous matter, which a cursory observer would certainly have supposed to be a natural deposit in the ground; but its history was known from some rubbish through which lime had filtered, when part of Buckingham House was built, and there were bricks, tiles, &c., underneath it.

he calls *M. Elephantoides*, because his teeth approach much nearer the elephant's than those of Cuvier's species, or of the Latidens. This animal appears to have been smaller than the elephant. A hippopotamus smaller than the living animal, a rhinoceros, a tapir, and others, have also been traced among these remains, as have a new lizard near the garial, and a crocodile near the common animal.*

The second of these discoveries was made on the north-east border of Bengal, at Carivari, near the Brahma-putra river. The remains were examined by Mr. Pentland. He traced a new species of anthracotherium, which he calls *Silisestre,* a new carnivorous animal of the weasel tribe, and a pachydermatous animal much smaller than any hitherto known, either living or fossil.†

The third and most remarkable of these collections is one discovered in the Markanda valley, and the Sivalik branch of the Himalaya mountains, in the year 1835. The curiosity of naturalists in India was immediately roused, and their industry directed towards the subject with that ardour which the relaxation of a sultry climate never abates, and that combined perseverance and ability which has

* Geol. Trans. N. S. vol. ii. pt. 3. † Ib.

ever marked the great men of our eastern settlements. Dr. Falconer and Captain Cautley have chiefly signalized themselves in this worthy pursuit; valuable aid has likewise been rendered by Lieut. Durand; and the result of their labours occupies one-half of the Asiatic Researches for 1836. They found first of all a new animal, of the ruminating class, whose skull is the size of a large elephant's, and which has two horns rising in a peculiar manner from between the orbits, with an orifice of great breadth and an extraordinary rising of the bones of the nose. They gave it the name of *Sivatherium*, from the place of its discovery, dedicated to the deity Siva. The breadth of the skull is twenty-two inches. Dr. Buckland has no doubt that it must have had a trunk, something intermediate between the elephant's and tapir's. They next found a hippopotamus of a new species, distinguished by having six incisive teeth, and a skull materially different from the other species, whether living or extinct. A new species of tiger was also discovered, which they called *Felis Cristata*, distinguished chiefly by the great height of the occipital bone. In the same place with these bones were found remains of the mastodon, and other known species of extinct animals; but the most interesting discovery was that of a camel, of which the skull and jaw were found. It is to be observed that no decisive proof of any of the Camelidæ, either camel, dromedary, or llama, had ever been hitherto found among fossil bones, although Cuvier had proved certain teeth brought from Siberia to be undoubtedly of this family, if they were really fossil, which he doubted. This discovery in India was therefore extremely interesting, as supplying a wanting genus. But for this very reason it became the more necessary to authenticate the position of this supposed camel's remains the more clearly, especially as there were abundance of existing camels in the country, which there could not be in Siberia. The Indian account is somewhat deficient in this respect, leaving us in doubt whether the bones, admitted to bear a very close resemblance to the living species, were found in a stratum or loose and detached.*

Beside all these additions to our knowledge of species and genera, two remarkable observations or sets of observations have been first made by osteolo-

* Asiatic Researches, vol. xix. pt. 1. Still more recently, it is said, a bone of the genus Junia has been found in the Sivalic Hills, and another in digging at Calcutta; but the particulars are unknown to me.

gists since the time of Cuvier. The one of these is the tracing of footsteps, the print of which has been left by animals upon the sand, or other material of the strata, while in a soft state. The other is Dr. Buckland's study of the intestines from their fossil contents, which he has called *coprolites*.* The first of these curious inquiries is conducted by observing the impressions which the softer and more destructible parts of animals, whose very race has been extinct for ages, made upon the earthy strata of a former world; it is the object of the other inquiry to ascertain from the petrified fæces bearing the impress of the alimentary canal, the internal structure of extinct animals; and both subjects are certainly calculated powerfully to arrest our attention.

The footsteps, it appears, were first observed by my reverend and learned friend, Dr. Duncan (to whom the country is also so deeply indebted as the author of savings' banks), in Dumfriesshire. On examining a sandstone quarry, where the strata lay one over the other, or rather against the other, for they had a dip of forty-five degrees, he found these prints not on one but on many successive layers of the stone; so that they must have been made at distant periods from each other, but when

* Κοπρος, fæces; λιθος, stone.

the strata were forming at the bottom of the sea. No bones whatever have been found in those quarries. Similar impressions, though of smaller animals, have been observed in the Forest marble beds near Bath. The marks found in Dumfriesshire, of which there were as many as twenty-four on a single slab, formed as it were a regular track with six distinct repetitions of each foot, the fore and hind feet having left different impressions and the marks of the claws being discernible. They appear to have been made by some animal of the tortoise kind.* But similar marks have since been found in other parts of the world. At Hessberg, in Saxony, they have been discovered in quarries of grey and red sandstone alternating; the marks are much larger than those in Scotland, and more distinct. In one the hind foot measures twelve inches in length, and the fore foot is always much smaller than the hind. From this circumstance and from the distance between the two being only fourteen inches, it is conjectured that the animal was a marsupial, like the kangaroo. But one of the most remarkable circumstances observed is, that the upper stratum has convex marks answering to the concavity of the lower slab on which it rests, clearly show-

* Edin. R. S. Trans. 1828.

ing that the former was deposited soft after the latter had been first printed by the foot in a soft state and then somewhat hardened. Dr. Kaup has termed the large unknown animal *Chirotherium*,† from the supposed resemblance of the four toes and turned out thumb to a hand. In the summer of 1838 similar footsteps of the chirotherium, and of four or five small lizards and tortoises, with petrified vegetables of a reedy kind, have been observed in the new red stone at Storeton Hill quarry in Cheshire, near Liverpool. A discovery has within the last two years, been made in the state of Connecticut, near Northampton, where the footsteps of various birds, differing exceedingly in size, are found in inclined strata of sandstone, and evidently made before it assumed its present position. The marks are always in pairs, and the tracks cross each other like those of ducks on the margin of a muddy pond. One is the length of fifteen or sixteen inches, and a feathery spur or appendage appears to have been attached to the heel, eight or nine inches long, for the purpose of enlarging the foot's surface, and, like a snow-shoe, prevent the animal's weight from sinking it too deep. The dis-

* Χειρ, hand.

tance between the steps is proportioned to their length, but in every case the pace appears to have been longer than that of the existing species of birds to which they approach nearest, the ostrich. Consequently, the animal must have been taller in proportion to his size. How much larger he was than the ostrich may be gathered from this, that the large African ostrich has only a foot of ten inches long, less than two-thirds of this bird, and yet stands nine feet high. These proportions would give a height of fourteen feet to the extinct animal. Some of the footsteps in the Storeton Hill quarry are eighteen inches in length. In the Forest marble of Bath the footmarks of small marine animals are descried.

In examining the inside of the ichthyosaurus, the half-digested bones of the animals on which these ravenous creatures preyed are found in large masses. But there are also scattered in great abundance among their fossil remains the fæces which they voided; and these being in a petrified state have preserved the very form of the intestines in minute detail. The fæcal matter is generally disposed in folds, wrapt round a central axis spirally. Some of these coprolites exhibit the appearance of contortion, and show that the intestines of the animal were spirally twisted; others, especially the smaller ones, give no such indications. The scales and bones of the prey are distinctly to be traced in the mass; these are the remains, undigested, of contemporary fishes and reptiles, including smaller ones of the beast's own tribe, on which he appears to have fed, as well as on other species. The light which these coprolites throw upon the structure of the animal's intestinal canal is sufficiently remarkable. The intestines are proved to have been formed like an Archimedes' screw, so that the aliment in passing through was exposed within the smallest space to the largest surface of absorbent vessels, and thus drained of all its juices, as we find in the digestive process of living animals. The similar structure of the intestinal canal in the sharks and dogfish now existing has been noticed by naturalists; and Dr. Paley expressly refers to it as making compensation by its spiral passage for its being straight, and consequently short, compared with the intestinal passage in other animals. We also can distinctly trace in these coprolites the size and form of the folds of the mucous membrane that lined the intestines, and of the vessels which ran along its surface. As there is no part of the animal frame more easily destructible than the mucous membrane and its vessels, the preservation of its casts is certainly a peculiar felicity for the physiologist. Similar observations have, since Dr. Buckland's discovery, been made upon the coprolites of fossil fishes, in the Lyme Regis lias, in Sussex, in Staffordshire, and near Edinburgh. In some places they take so fine a polish that lapidaries have used them for cutting into ornamental wares. One of the most singular coprolites was found by Lord Greenock (an assiduous and successful cultivator of natural science) between the laminæ of a block of coal near Edinburgh, and surrounded with the scales of a fish recognised by Professor Agassiz as of contemporary origin. To these observations a very curious addition has been made by the Professor, who found that the worm-like bodies described by Count Munster, in the lithographic slate of Solenhofen, are in fact the petrified intestines of fishes, and he has also found the same tortuous bodies occupying their ordinary position between the ribs in some fossil remains. He has named them *Coleolites;** and certainly the representation given of them in

* Κωλον, the great intestine.

the drawing resembles an actual intestine as accurately as if it were the portrait of it.

When Cuvier abandoned to Professor Agassiz the whole department of Fossil Ichthyology he showed as happy and just a discernment of living character as he ever displayed in the arrangement and appropriation of animal remains. That admirable person has amply earned the honour thus bestowed on him by devoting his life to this extensive, obscure, and difficult study. The results of his laborious researches have been from time to time published in a great work upon fossil fishes; but as the arrangement followed as yet in the publication necessarily leaves the several parts incomplete, a distinct and satisfactory view of the whole cannot be formed until the work is finished. Some of the discoveries, however, which bear upon the subject of our present inquiries may be shortly described. The importance of the study to fossil geology is manifest from this, that the class of fishes being continued through the successive periods of the different formations, while those of land animals are confined each within certain limits, and the fishes being also inhabitants of those waters in which all the aqueous deposits once were contained, we are enabled by Fossil Ichthyology, through various periods of the earth's formation, to pursue the comparison of a vertebrated animal's condition in each stage.

The Professor's classification is founded upon the form of the scales, which are adapted to the structure of each tribe, and afford a perfectly scientific principle of arrangement. He thus divides the whole into four orders:—the *Placoideans** whose scales are irregular enamel plates more frequently broad, but varying in dimensions down to a point or prickle; the *Ganoideans*† with angular scales of bone or horn thickly enamelled and shining; the *Ctenoideans*‡ with comb-like scales having a jagged edge and no enamel; § the *Cycloideans*,|| whose scales are smooth at the edge and composed of horn and bone, but unenamelled.¶

There were in all 8000 species of fish enumerated by Cuvier, of which more than three-fourths, or 6000, belong to the two last classes, and no one of either of these classes has ever been found in any formation anterior to the chalk; so that the whole of these 6000 kinds of fish have, to all appearance, been called into existence at a period long after the

* Πλαξ, a tablet or plate. † Γανος, brilliancy.
‡ Κτεις, a comb. § Perch belong to this class.
|| Κυκλος, a circle. ¶ Salmon and herring are of this class.

primitive, the transition, and all but the latest secondary formations. On the other hand, and in the earlier times of the secondary and transition strata, there existed species of the other two orders, which have comparatively few representatives surviving to our days. The Professor has thoroughly examined 800 fossil species of these two orders, and finds not a single exception to the rule thus laid down for the relation between different species of animals and successive formations of strata.* His deductions received further corroboration by the examination of 250 species, all of new and extinct fishes, submitted to him in England, and which were, for the most part, found in this country. The analogy in this respect between the results of Fossil Ichthyology and those of Cuvier's Researches is striking throughout. In the lower deposits of the lias there are found the remains of the great sauroid fishes analogous to the fossil lizards of the same strata. More than two-thirds of the fishes found in the chalk strata are of genera now extinct. These extinct genera, however, of the newest secondary strata approach more nearly to the fishes of the tertiary strata than the fishes found in the oolite or Jurassick formation; insomuch that the

* Rapport sur les Poissons Fossiles, 1835, p. 38.

professor is disposed to range the chalk and greensand nearer to the tertiary than secondary formations on this account. Not a single genus even of those whose species are found in the Jurassick deposits is now known among existing fishes; nor is there a single species, and but few genera common to the chalk, and the older tertiary strata. A third of those found in the strata of the later tertiary formation, as the London clay and the coarse limestone of the Paris Basin, are of extinct genera. The Norfolk crag and upper sub-appennine formation have, for the most part, genera found in the tropical seas; the tertiary formation generally approaches nearest to our living species, but the Professor affirms that, except one small fish, found in modern concretions on the coast of Greenland, not a single species exactly the same with those of our seas is to be found in a petrified state. This continued analogy is very important in a geological view.

In a zoological view it would be endless to attempt any analysis of the Professor's researches. Among the extinct species no less than 150 belonged to the family of sharks, whose services, in keeping down the increase naturally so rapid of fishes, have been required in all ages of the ocean.

Different kinds of shark, however, appear to have belonged to different periods. Of the three sub-families into which the Professor divides the great class of sharks, the first is found in the earliest period of organic remains, the transition strata, and continues till the beginning of the tertiary, but there is now only one species of it existing, and that is found in New Holland. The second sub-family begins probably with the coal formation, and ceases when the chalk commences. The third begins with the chalk, and continues down through the tertiary formation to the present time. The form as well as the size of the extinct species differ in most things materially from the living, and in no respect do they vary more than in their covering or scales.

As the coprolites enable us to ascertain the interior structure of the extinct reptiles, so do they throw light upon that of fishes also, those especially of the sauroid or lizard-like kind. We have even instances of their intestines being partially preserved by some fortunate accident. An example near Solenhofen has been mentioned already. A specimen was found in Sussex, where the stomach, with its different membranes, was retained. In a number of fishes found in the Isle of Sheppy the bony capsule of the eye was found entire; and in some other instances the plates forming the gills or branchiæ are perceivable.

It thus appears that great and important additions have been made to this interesting science since Cuvier, who may properly be termed its founder, ceased from his labours. But it would not be proper to pass from a consideration of the services rendered by his successors, without making mention of one illustrious inquirer, a man of truly original genius, who preceded him by a few years. John Hunter, whose unrivalled sagacity seemed destined to cast a strong light upon whatever walk of science he trode, had turned his attention, as early as 1793, to fossil bones, in consequence of a collection sent to this country by the Margrave of Anspach. He described and commented upon them in detail with his wonted acuteness; he adopted the same safe and natural course which Cuvier afterwards pursued with such signal success, of examining the known bones of existing species as well as those submitted to his consideration; and it appears, from some of his concluding remarks, that he perceived distinctly enough the specific difference of the fossil animals, at least of some among them. Thus, having compared the fossil skull of a supposed bear with that of a white bear which he had procured from the owner of the animal while alive, he gives an accurate drawing of both, and marks their diversities, indicating his opinion that the fossil animal differed from all known carnivorous animals.* Who does not perceive that he was on the right track, and would have reaped a plentiful harvest of discovery, had he devoted himself to the general investigation of the subject?†

II. The speculations of succeeding zoologists or comparative physiologists have not only made no impression upon the anatomical results of Cuvier's inquiries, but they never appear to have been pointed towards that object. Considering the numberless instances in which he had to draw his conclusions or to form his conjectures from a very imperfect collection of facts, it is wonderful how constantly the fuller materials of his followers have confirmed his inferences. But geological inquirers

* Phil. Trans. 1794, p. 411.

† In the Hunterian Museum there is a large collection of fossil organic remains, selected with consummate skill, and showing the attention bestowed by this great man on the most delicate parts of organization which they exemplify.

have occasionally impugned his doctrines respecting the relation of the classes of animals to the successive formations of the strata that incrust our globe. It has been denied by some that any such relation at all can be truly said to exist. There seems, however, no possibility of maintaining this position, whether we agree wholly with Cuvier or not in the detail of his statements. For the fact is undeniable that some strata, let them have been arranged in whatever succession, formed and placed by whatever causes, contain the remains of certain classes of animals which are not to be found in other strata. It is another fact equally indisputable, that no animals now exist of the same kind with the greater part of those found in any of the strata. This appears to connect the different races of animals with the different strata. But it is said that this is not a chronological connexion, and affords no evidence of strata having been formed rather in one age than another. If it were so, there still would remain a foundation for the position which merely affirms a relation between organic remains and strata. But is it true? The principal reason assigned is, that although no animals of a certain kind are found in certain strata, supposing those strata to have been formed at a given period, the animals of the kind in question may have perished so as not to have been washed into the sea or other water in which the earthy matter was mixed, and from which it was deposited. Now, not to mention that this bare possibility becomes improbable in the degree in which the facts are multiplied and the observations of animals and strata extended, the researches respecting fossil fishes seem to negative the objection entirely. For if the different strata were made by the sea, and contain totally different remains of marine animals, it is clear that each must have been formed respectively in a sea inhabited by different animal tribes. The strict parallelism, too, which is observed between the connexion of different races of animals and that of fishes with different strata, lends the strongest confirmation to Cuvier's doctrines.

Ingenious and laborious attempts have been made to show, that though many races of animals are now wholly extinct, the evidence fails to prove the non-existence of any race (except our own) at a preceding period; in other words, to disprove the proposition that many of the present races came for the first time into existence at a period subsequent to the time when we know that others existed, always excepting the human race, which it is admitted we have sufficient reason to believe did not exist in the earlier stages of the globe's formation. It cannot, however, be denied, *first*, that the extinction of many races of animals, which is admitted, affords a ground of itself for thinking it probable that new ones should be found to supply their places; *secondly*, that there seems nearly as little reason to regard the utter extinction of some classes as more improbable than the formation of others; *thirdly*, that the admitted creation of man destroys the whole support which the objection might derive from a supposed uniformity of natural causes, always acting, and removes the difficulty said to exist, of assuming different sets of principles to be in action at different periods of the world; *fourthly*, that the great number of facts which have been observed, all pointing uniformly in one direction, cannot be got over by suggesting mere possibilities for explanations. The improbability is extreme of one set of animals having existed at the same age with another set, when we find certain strata having the traces of the former without any of the latter, and *vice versâ*. This improbability increases in proportion to the number of the species. If these exceed hundreds, and even amount to many thousands, the improbability becomes so great as to reach what, in common language, we term a moral impossibility. Now, there are 6,000 kinds of fishes, of which not one specimen is to be found in any of the formations preceding the chalk. But suppose we lay out of view all question of one formation being older than another, there are certain strata in which none of those species are found. There is no disposition to deny that these strata were formed in the water; therefore, at whatever time they were suspended in the water, that water at that time contained none of those 6,000 kinds which now people it. Then from whence did they all come if they existed at that period, and yet were not in the water when the strata were formed? But it is equally admitted that the water in those days contained many other kinds of fish now extinct, and found only in certain strata, and it contained some few which we find in other strata, and some which are still to be found in the sea. Can anything be more gratuitous than to suppose that all the fishes of a certain class were destroyed at the formation of those strata, while all those of another class were afterwards brought from a different part of the sea to succeed the last ones, and a certain small number survived to mix with other strata, or even to last till now?

The only sound objection that can be taken to the theory, is that to which the absolute assertion of the fact is liable. We can easily ascertain that certain species are no longer to be found living on the globe But we may not be so well able to affirm with certainty that certain fossil genera of one formation may not hereafter be found in another, or, which is the same fact in another form, that certain living species may not be traced among fossil remains. Thus the small family of the camel was wanting in all our fossil collections till the late discoveries in the Himalaya mountains have made it probable that a species of this class may be found to have existed there with the mastodon and other extinct mammalia. This is possible, perhaps likely. So an ape's jaw is supposed for the first time to have been found in a fossil bed in France with other races, and no quadrumane had ever been before traced in any part of the fossil world. The proof of this discovery is, however, as yet, involved in some doubt, and even were it more precise, we should only have two instances in which the negative evidence had failed, leaving a multitude of others, hundreds of land and thousands of sea animals, of which no representatives are to be traced among the fossil remains of any country. It must always be recollected that the whole argument rests upon probability, more or less high. Even as regards the admitted non-existence of the human species, the mere evidence of osteological researches is not demonstrative; for although it is quite certain that among the thousands of animal remains which have been discovered and carefully examined, not a fragment of a human bone is to be found, it is barely possible that in some deposits as yet unexplored the skeleton of a man may be discovered. We have at present only to make our inference square with the facts; to affirm that, as far as our knowledge extends, there is no such relic of our race in the earlier strata of the globe; and to conclude that, considering the extent of past inquiries, the regularity of the connexion between other races of different kinds and various strata, and the portions of the earth over which our researches have been carried, the very strong presumption is against any such contradictory discovery being hereafter made.

III. Whatever opinion men may form upon the question raised by some antagonists of Cuvier's geological doctrines, all must allow that considerable light has been thrown upon the subject of discussion by their labours. Indeed a considerable addition to our knowledge has been made by some of these able and learned men, even admitting that they have failed to impugn the theory, and taking the facts which they have ascertained as forming an addition, by no means inconsistent with it. Thus the valuable work of Mr. Lyell has, in two essential respects, greatly advanced geological knowledge. He has examined, with a much more minute attention than had ever before been given to the subject, the action of the physical agents actually at work before our eyes, and has shown how extensively these may operate upon the structure of the earth's surface. It may be admitted, perhaps, that Cuvier had somewhat underrated their power, although the reader may still retain his opinion, that the force ascribed from the facts to those ordinary physical powers is inadequate to produce the effects which the phenomena present; that all the violent and sudden actions known on the globe are topical, being confined within comparatively narrow limits, and that the supposition of sudden and even instantaneous change on a vast scale in former periods has been too lightly taken up. Indeed, unless we suppose such changes as might happen from the disruption of a continent united by a small neck of land, like that which may be found once to have joined Gibraltar and Ceuta, it seems hard to imagine how a tract of country, extending from Holland to beyond the Caspian, and from Scandinavia to the Carpathian mountains, could be drained of the sea, which certainly once covered it, or, having still more anciently been dry, could have been laid under water.*

But a much more important service has been rendered by Mr. Lyell's comparison between the different formations of the tertiary class; and although it is with unavoidable distrust of himself that any one little versed in geological science should venture to speak, it should seem that the division which he has thus succeeded in tracing of the tertiary period, may stand well with the previous system of Cuvier, and be received as a fact independent of the controverted matter with which it has been connected. With the important aid of several eminent conchologists, but especially of Mr.

* In Mr. Whewell's learned work on the History of the Inductive Sciences, there are some acute and important remarks on the two theories, that of Uniform Action, and that of Catastrophes. B. xviii. c. 8.

Deshayes, he examined the numbers of testaceous animals traced in different formations; and finding that in some strata the proportion of shells of living species was very different from others, he distributed the strata of this tertiary period into three classes accordingly; the earliest being those which contained the fewest of our living species. The latest of the three periods into which he thus subdivides the tertiary era he calls *pliocene,** or more recent; the next before *miocene,*† or less recent; the earliest *eocene,*‡ or dawning. Seventeen species of shells are common to the three divisions, of which thirteen still exist and four are extinct. In the pliocene the proportion of existing shells always exceeds one-third, and usually approaches one-half of the whole found. In the miocene, the existing shells fall considerably short of one-half, that is, the extinct species preponderate; indeed, of 1021 examined, less than a fifth were existing. There are 196 common to this and the last period, of which 82 are extinct. In the eocene period, the proportion of existing shells is much smaller, not exceeding three and a half per cent.; and there are

* Πλειων, more, and καινος, recent.
† Μειων, less. ‡ Ηως, dawn.

only 42 common to this and the miocene. In the Paris Basin 1122 species have been found, of which only 38 are now known as living.

The theory of Cuvier and Brongnart respecting the successive formations in the Paris Basin, appears to require some modification in consequence of more recent examination. They considered that upon the chalk there was laid, first, a fresh-water formation of clay, lignite, and sandstone; then a marine formation of coarse limestone; and then upon that a second fresh-water formation of silicious limestone, gypsum, and marl. The researches of Mr. Constant Prevost seem to show that instead of these three successive formations, there were laid on the chalk a clay formation of fresh-water origin, and then upon that, contemporaneously, three others, in different parts of the same Basin, namely, a fresh-water formation of silicious limestone, another of gypsum, and a marine formation of coarse limestone. In the rest of the series the two theories coincide.

It must, however, be observed that the more important doctrines of Fossil Osteology, even as regards their connection with the history and structure of the globe, do not necessarily depend upon the opinions which may be entertained of the more controverted points of geological theory, while the science of comparative anatomy exists alone, self-contained, and independent of geology. But all must agree in admitting the important service which Osteology has rendered to geological inquiries, and in rejoicing at the influence which it has had upon those who pursue such speculations, in promoting a more careful study of facts, and recommending a wise postponement of theoretical reasoning, until the season arrives when a sufficient foundation for induction shall have been laid by the patient observer.

NOTES TO THE FOSSIL OSTEOLOGY.

NOTE I.

As some learned men are satisfied with the proofs of an ape's jaw-bone having been found at Sansan, in the south-west of France, and an astragalus of the same genus in the Sivalick hills, it is very possible that this genus may be added to those found in the strata of the Miocene period; for it is only in the more recent formations that these remains are supposed to exist. That they should be found in any of the Pliocene formations is in a high degree improbable; and even then we have only got to the middle of the Tertiary period. No one contends that in the earlier formations any such remains are to be traced.

But in case any objection should be raised to the argument in the text, upon the supposition that, because quadrumanous animals were supposed by Cuvier not to be traceable in any but the pre sent portions of the globe's crust, therefore human remains may likewise hereafter be found in earlier formations, we may remark that, even if they were, contrary to every probability there found, no one pretends to expect such remains in those strata where no mammalia of any kind have been discovered; and the argument in the text is wholly independent of the particular period at which the non-existence of our race is admitted. These considerations are fit to be borne in mind, since learned men, like Mr. Schmerling, are inclined to think that some human bones found in the same caves with the remains of hyænas and other animals are of contemporaneous origin. The great majority of geologists, however, refer the animals in question to the last geological era before the creation of man.

NOTE II.

The state of rapid and solid advancement in which the science of Palæontology now is, may make the summary of its doctrines in any one year little applicable to the next. The notes to the Analysis of Cuvier, and the subsequent account of the labours of his successors, may serve to show what inhabitants of the former surface of the earth are at present within our knowledge. But with respect to the two important classes of ichthyosaurus and plesiosaurus, the following abstract will prove convenient to the student who would compare the present state of our information upon these two fossil genera at present with what it was when Cuvier wrote. Nothing can better exhibit the rate, as it were, at which this science has been advancing. I am indebted to my learned, able, and excellent friend, Mr. Greenough, for this summary, which will be found to be marked with the accuracy, the clearness, and the conciseness which distinguish all his productions :—

ICHTHYOSAURUS.

1. Communis ....Cuvier, vol. ii. Lias—England and Wurtemberg.
2. Coniformis ....(See Journal of Acad. of Philadelphia.) Not known to Cuvier. Lias—Bath.
3. Grandipes.....(Geol. Proc., 1830.) Not known to Cuvier.
4. Intermedius ...Lias—England and Wurtemberg.
5. Platyodon.....Lias—England and Wurtemberg.
6. Tenuirostris ...Lias—England and Wurtemberg.
7. Ichthyosaurus .Kimmeridge clay.
8. Ichthyosaurus .Muschelkalk—Luneville and Mannsfeld.

PLESIOSAURUS.

1. Goldfussii.....Quarries of Solenhofen. Not known to Cuvier.
2. Carinatus .....Lias—England and Boulogne.

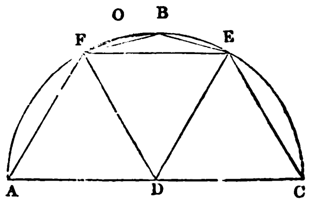
3. Dolichodeirus .Muschelkalk—Germany; and lias—England.
4. Pentagonus ...Jura beds—France.
5. Profundus.....Variegated sandstone—Jura. Not known to Cuvier.
6. Recentior .....Kimmeridge clay.
7. Trigonus ......Calvados—north of France.
8. Trigonus......Cuvier, vol. ii. p. 486. Lias, probably.

PRINCIPIA.

THIS work is justly considered by all men as the greatest of the monuments of human genius. It contains the exposition of the laws of motion in all its varieties, whether in free space or in resisting media, and of the action exerted by the masses or the particles of matter upon each other, demonstrated by synthetic reasoning; and it unfolds the most magnificent discovery that was ever made by man—the Principle of Universal Gravitation, by which the system of the universe is governed under the superintendence of its Divine Maker. Two of the three Books into which the treatise is divided are chiefly composed of mathematical investigations, conducted by the most refined and profound, but at the same time the most elegant application of geometry and of a calculus which is only a particular form of the fluxionary method invented by the illustrious author in his early years. The third Book contains an explanation of the motions of the heavenly bodies, deduced chiefly from the first portion of the former part, and grounded upon the phænomena observed by astronomers. This concluding portion, however, of the great work, is also interspersed with geometrical reasoning of the same admirable description as characterized the former, and applied to the solution of problems respecting the heavenly motions.

Before Sir Isaac Newton appeared to enlighten mankind, and to found a new era in the history of physical science, the eminent men who had preceded him had made, during the century immediately preceding his birth, very important steps in furthering the advancement of our knowledge; and they had approached exceedingly near that point which forms the most important of all his discoveries, according to a kind of law which seems to regulate the progress of human improvement—a law of continuity, which apparently prevents any sudden, and, as it were, violent change, from being made in the intellectual condition of the species, and prescribes the unfolding of all great truths by slow degrees, each mighty discovery being preceded by others only less considerable than itself, and conducting towards it. The great discoveries in pure mathematics afford striking examples of this truth. That of logarithms by Lord Napier is, perhaps, the instance in which the most considerable deviation has been made from the rule; but even here there had been some curious methods of mechanical calculation invented before, and the discoverer of logarithms himself had reached the point very nearly by other most ingenious contrivances, before he actually made his great step.

But the fluxionary or differential calculus gives a remarkable exemplification of the general principle; and its subsequent most important extension, the calculus of variations, furnishes another not less striking. Before Newton and Leibnitz made the grand discovery from which all the progress of the moderns, in mixed as well as in pure mathematics, has been derived, ever since Descartes opened the way by his happy application of algebra to geometry, mathematicians had been intent upon the resolution of problems connected with the rectification and quadrature of curves, and the determination of points that possess properties of maxima and minima, as well as the finding of normals, tangents, and osculating circles. These inquiries had led them to consider the laws by which the relations between the ordinates and abscissæ referred to any given axis are governed at different points of that axis; for in truth that implies the nature of the curvature itself, and includes the manner in which the length of the curve line increases or diminishes, as well as the space which it incloses. They were thus led to examine the generation of those curve lines and curvilinear spaces, whether that is conceived to be effected by the movement in the one case of points, and in the other of straight lines, or is supposed to be produced by the constant juxtaposition of indefinitely small straight lines inclined to each other according to a given law, in the one case, and indefinitely small rectangles in the other. The latter is perhaps the more natural supposition of the two, and not the less easy. For if any one is set to measure the area of a field bounded by a curvilinear outline, as he can at once measure a space inclosed within straight lines, his course will be to divide the given space into squares (or rectangles, if it be an oblong), and then to divide each of the smaller curvilinear spaces into other rectangles, and so on till he has exhausted the whole by a series of rectangles, always decreasing in size as they increase in number, and the last of which seem to coincide nearly or sensibly with the area of the outer or curved line of boundary. Thus he would proceed by trial and actual measurement of the space; and thus do land measurers (the lineal descendants of the first geometers, as well as their namesakes) still proceed. But speculative mathematicians being aware of the general properties of the lines they have to examine, and these being regularly formed, which the boundary of the field is not, they could calculate the relations to each other of the sides of the rectangles into which they divided the figure, and thus form series of rectilinear figures diminishing in size, and which series might be carried to any length so as ultimately to exhaust the curvilinear area. Thus A B C being a semi-



circle, it was easy to find the area of the semihexagon or three equilateral triangles A D F, F D E, and D E C, and then of the triangles F B E × 3, and again of the triangles F O B × 6, and so on; so that the radius A D being called r, there was obtained a series of this form, $\frac{3}{2} r^2 \sqrt{3} + \frac{3}{2} r^2 (2 - \sqrt{3}) +$

$$\frac{3 r^2}{2} \left(\frac{2 \sqrt{3} - 2 - \sqrt{2}}{2} \right) + \text{\&c.,}$$ and in like manner an approximation to the length of the circle may be found.

But the extreme cumbrousness of this calculus, which is still more unmanageable in other curves where the radii are not, as in the case of the circle, equal, made it necessary to find some other method; and geometricians accordingly examined the laws by which the areas increase in each curve, so that by adding all those innumerable increments together their sum might give the exact space required. The same process was attempted with the lengths of the curves, considering them as polygons whose sides diminished while their numbers increased indefinitely. In this way Cavalleri, Fermat, and Wallis, and still more Harriot and Roberval, appear to have come exceedingly near the discovery of the general rule for performing these operations before Newton and Leibnitz, unknown to each other, made the great step. Roberval especially had solved many problems of quadrature and of draw ing tangents, by methods extremely similar to the Newtonian; nor were the ancient methods of exhaustion and indivisibles so far distant as to let us doubt that, had the old geometers been possessed of the great instrument of algebra, and bethought them of its truly felicitous application according to the idea of Descartes, long before our times they would have anticipated the discoveries which form the great glory of modern science.

The discovery of the Calculus of Variations affords a similar example of gradual progress. When the differential calculus had enabled us to ascertain the maxima and minima of quantities, for example the value of one co-ordinate to a curve, at which the other becomes a maximum or a minimum, or which is the same thing, the point of greatest and least distance between the curve and a given right line, or which is the same thing, when the general relation of the co-ordinates being given we were enabled by means of the calculus to examine what that particular value was at which a maximum property belonged to one of them—then geometricians next inquired into the maxima and minima of different curves, that is to say, into the general relation between the co-ordinates which gave to every portion of the curve a maximum or a minimum value in some respect. Thus, instead of inquiring at what value of x (the abscissa) in a known equation between x and the ordinate y, y became a minimum, or the curve approached the nearest to its axis, the question was what relation x must have to y (or what must be the equation as yet unknown) in order to make the whole curve, for example, of the shortest length between two given points, or inclose with two given lines the largest space, or (having some property given) inclose within itself the largest space, or be traversed in the shortest possible time by a body impelled by a given force between two given points. Here the ordinary resources of the differential calculus failed us, because that calculus only enabled us, by substituting in the differential equation the value of one co-ordinate in terms of the other, to make the whole equal to nothing, as it must be at the maximum or minimum point where there is no further increase or decrease. But here no means were afforded of making this substitution, and the problem seemed, as far as this method went, indeterminate. Various very ingenious resources were employed by Sir Isaac Newton, who in the Principia seems to have first solved a problem of the Isoperimetrical class—that is, finding e solid of least resistance; and soon after by the Bernouillis and other continental mathematicians, who worked by skilful constructions and suppositions consistent with the data. The calculus called that of Variations has since been invented for the general solution of these and other similar problems. It consists in treating the relations of quantities, or of their functions, as themselves varying, but varying according to prescribed rules, just as the differential calculus regards the quantities themselves, or their functions, as varying according to prescribed rules. It bears to the differential calculus somewhat of the relation which that bears to the calculus of fixed and finite or unvarying quantities. It is wonderful how very near Bernouilli, when he solved the problem of finding the line of swiftest descent, came to finding out the calculus of variations; if, indeed, he may not be said to have actually employed it when he supposed, not as in the case of the differential calculus, two ordinates of a known curve infinitely near one another, but three ordinates infinitely near, including two branches of an unknown curve, each infinitely small; for he certainly made the relation of these ordinates to the abscissa vary. Euler used the calculus more systematically in the solution of various problems; but he was much impeded for want of an algorithm. This important defect was supplied by Lagrange, who reduced the method to a system and laid down its general principles; but had Euler gone on a little step further, or had Bernouilli been bent on finding out a general method instead of solving particular problems, or had Emerson, who has one or two similar investigations in his book on Fluxions, reduced the method by which he worked them to a system by giving one general rule (which, writing a book on the subject, he was very likely to have done), the fame of that discovery would have been theirs, which now redounds so greatly and so justly to the glory of Lagrange.

The discovery of Gravitation as the governing principle of the heavenly motions is no exception to the rule which we have stated of continuity or gradual progress. When Copernicus had first clearly stated the truth to which near approaches had been made by his predecessors, from Pythagoras downwards, that the planets move round the sun, and that the earth also moves on its axis while the moon revolves round the earth, he yet accompanied his statement with so little proof beyond the agreement with the phenomena, which the Ptolemaic hypothesis could equally boast of,* that for more than half a century afterwards it had no general acceptance, Bacon himself rejecting it; when Galileo, by his telescopic discoveries, especially of the phases of Venus and the satellites of Jupiter, and by his yet more important discoveries in the laws of motion, may be said first to have proved the truth of the Copernican system. Afterwards the satellites of Saturn, added to Kepler's observation of Mercury's transit over the sun, afforded most important confirmation. The great discoveries of this eminent man followed close after those of Galileo; the motions of the planets were found to be in ellipses with the sun in one focus; lines drawn to the sun from them were found to describe areas proportional to the times of their revolution; and the relation was established between the squares of those times and the cubes of the distances of the bodies from the focus.

* It is certain that its greater simplicity was, before Galileo's time, the only argument in favour of the Copernican theory against the Ptolemæan.

How near this brought scientific men to the cause or law of the whole is manifest, especially when we regard the connexion thus established between the revolving bodies and the great luminary in the centre. Although Kepler himself erroneously mingled with the influence which this law of motion led him to ascribe to the sun a transverse force which he deemed necessary to maintain the projectile motion of the planets round the centre, yet others formed more correct ideas of the matter. It seems to have been Huygens, who, fourteen years before the "Principia" was published, first showed the true nature of centrifugal forces; but several years earlier, Borelli, in treating of the motion of Jupiter's satellites, considers the planets as having a tendency to resile from the sun and the satellites from the planets, but as being "drawn towards and held by those central bodies, and so compelled to follow them in continued revolutions." He also most accurately compares the receding (or centrifugal) force with the tendency of a stone whirled in a sling to fly off at every instant of its motion. Hooke, a man of unquestionable genius, and whose partial anticipations of many great discoveries are truly remarkable, about the same time with Borelli, asserted that the attraction of the sun draws away the planets from moving in straight lines, and that the force of the attraction varies with the distance. He had, as early as 1666, read to the Royal Society a paper explaining the curvilinear motion of the planets by attraction; and Halley, as well as others, had even hit upon the inverse duplicate ratio, by supposing that the influence from the sun was diffused in a circle, and that therefore the areas proportioned to that influence were as the squares of the radii, and consequently the intensities, being inversely as those areas, were inversely as the squares of the radii or distances. Finally, Hooke had foretold, that whoever set himself to investigate the subject experimentally would discover the true cause of all the heavenly motions.

Such were the near approaches which had been made to the law of Gravitation before its final and complete discovery. But although in this gradual progress it resembles almost all the other great improvements in science, in one material respect it differs from them all. The theory was perfect which Newton delivered, and the whole subject was at once thoroughly investigated; it was not that the general principle hitherto anxiously sought for, and of which others had caught many glimpses, was now unfolded and established upon appropriate foundations, but almost every consequence and application of it was either traced, or plainly sketched out; it was pursued into all the details; a systematic account of its operation was given, symmetrical, and in its main branches complete; so that, however nearly former inquirers had approached the general law, the distance was prodigious between their conjectures, how learned and happy soever, and the magnificent work which the genius of Newton had accomplished.*

It must be observed, too, that, beside this grand achievement, the Principia performed three other most important services to physical and mathematical science. *First.* It laid a deep and solid foundation for subsequent discoveries in the science

* The subsequent discoveries of mathematicians by means of the improvements in the calculus, have added new illustrations, and traced further consequences of the theory. But there is only one of their improvements which can justly be said to have advanced the evidence of the fundamental principle further than Sir I. Newton had carried it, by supplying any defect which he had left; we allude to the reconcilement by Clairaut of the moon's apogeal motion according to the theory with the observation. This is fully explained in the sequel. It forms one of the most interesting passages in the whole history of science.

of physical astronomy, both in the general principles of dynamics which it unfolded, and in the application which it made of these to the heavenly bodies and their motions. *Secondly.* It gave a complete system of dynamics applicable to all subjects connected with motion and force and statics—a system throughout abounding in the most important original mathematical truths, expounded and proved with singular beauty, though with extreme conciseness. *Thirdly.* It propounded and showed the application of a new calculus, or method of mathematical investigation, that method by the help of which those truths had been discovered; and others, before resting upon an empirical foundation, were demonstrated. Thus it is no exaggeration to say that, even if the great discovery of the law which governs the universe were taken away from the Principia, it would still retain its rank at the head of all the works of mathematicians, as the most wonderful series of discoveries in geometrical science, and its application to the principles of dynamics.

That the reception of this work was not such as might have been expected has frequently been alleged; and although an ingenious and well-meant attempt has lately been made by an eminent mathematician* to relieve this country from its share of the imputation, chiefly by showing the estimation the author was held in immediately after its publication, it is, on the one hand, certain that Newton's previous fame was great by former discoveries, and that after its appearance the Principia was more admired than studied. There is no getting over the inference on this head which arises from the dates of the two first editions; there elapsed an interval of no less than twenty-seven years between them; and although Cotes speaks of the copies having become scarce and in very great demand when the second edition appeared in 1713, yet had this urgent demand been of many years continuance, this reprinting could never have been so long delayed; nor was the next edition required for thirteen years after the second; so that in forty years the greatest work ever composed by man reached only a third edition, and that third has, during the succeeding hundred years, been the one generally in use, although translations and excerpts have been published from time to time, and two editions were printed on the Continent, one at

* Whewell's History of the Inductive Sciences, vol. ii.

Amsterdam and one at Cologne. The doctrines of the work were, however, much more readily embraced and more generally diffused in this country, which had the benefit of Maclaurin's admirable view of the more general principles of the system, published about the middle of the last century. On the Continent they made their way far more slowly; nor was it until Voltaire employed his great powers of clear apprehension and lucid statement to give them currency that the Cartesian prejudices of our neighbours gave way, and the true doctrine found a general and a willing acceptance.

It must be admitted that the manner in which the truths of the Principia were unfolded has added somewhat to the slowness of the world at large in embracing them, but greatly to the reluctance with which men have generally undertaken the task of reading that great work, and satisfying themselves of the proofs upon which its doctrines rest. Conciseness is everywhere rigorously studied; not only does the author avoid all needless prolixity and repetition in unfolding his discoveries, but he leaves out so many of the steps of his demonstration, and assumes his reader to be so expert a geometrician, that the labour of following him is often sufficient to deter ordinary students from making the effort. If mathematical reading is never the same passive kind of operation with other studies, the perusal of the Principia is emphatically an active exercise of the mind; for what appeared to the intuitive glance of him who could discover the theorem or solve the problem too plain to require any proof, may well stop common minds in their progress towards the point whither he is guiding them; the distances which he can stride at once over this difficult path must, by weaker persons, be divided into many portions, and travelled by successive steps. Add to which, that, as the method of proof is throughout synthetical, and as it is geometrical, the helps of modern analysis are thus withheld. Upon the whole, therefore, a most valuable service was rendered to students by the able and learned commentary of the Jesuits, Le Seur and Jacquier, who, in 1739 and 1742, published the Principia, with very copious illustrations, although it is to be regretted that they resort far less frequently to analysis than was desirable. It is remarkable enough, and affords an additional proof of the slow progress which truth had then made in some parts of Europe, that these excellent authors deemed it necessary to accompany their publication of the Third book, which treats of the heavenly motions, with a declaration in these words: "Newtonus in hoc tertio libro Telluris motæ hypothesim asserit. Autoris propositiones aliter explicari non poterant, nisi eâdem quoque facta hypothesi. Hinc alienam coacti sumus gerere personam; cæterum latis a summis Pontificibus contra Telluris motum Decretis nos obsequi profitemur." This edition is dated, as might be supposed, at Rome.*

The Principia begins with a definition of terms, and a compendious statement of the science of dynamics as it existed previous to Newton's discoveries. The definitions, eight in number, comprise that of *quantity of matter,* which is in the proportion of its bulk and density, the density being the proportion of its mass to its bulk—the *quantity of motion,* which is in proportion to the velocity and quantity of matter jointly—the *vis inertiæ,* which

* It must, however, be observed, that such bigotry and intolerance was not confined to Rome. As late as 1769, Buffon was compelled, by the interference of the Sorbonne, to publish a recantation of some portion of his fantastical theory of the earth, comprehending, as it happened, the very few things in it which had any reasonable foundation. We ought also to mention, for the credit of the Papal Government, that a late pontiff (Pius VII.) procured a repeal of the decree against the Copernican system

is the force or power of matter to persist in any given state, whether of rest or of motion in a straight line, and to resist any external force impressed upon it to change that state—*centripetal force,* which is the power that draws towards a given point or centre bodies at a distance from it—finally, the three kinds of amount of centripetal force; the *absolute* amount, in proportion to the intensity of the power exerted in drawing towards the centre; the *accelerating,* in proportion to the velocity generated in a given time; and the *moving,* in proportion to the motion generated in a given time towards the centre.*

Two things are worthy of remark in these definitions: *first,* that, as if foreseeing the cavils to which his doctrines would give rise, he guards, in a scholium to the definitions, against the supposition that he means to give any opinion as to the nature or cause of centripetal force, much less that he ascribes any virtue of attraction to mere centres or mathematical points; whereas he only means to express certain known and observed facts:

* There are eight definitions in the book, though we have only given them under seven heads, not having made a separate definition of the *force impressed,* which is here mentioned under the important head of the *vis inertiæ.*

secondly, that, in illustrating his definition of centripetal forces, he really anticipates his great discovery; for, after giving the examples of magnetic action, and of a stone whirled in a sling, he proceeds to the motion of projectiles, and shows how, by increasing the centrifugal force, they may be made to move round the earth, as may also, he says, the moon, if she be a heavy body, or in any other way be deflected towards the earth, and retained in her orbit. That force, he adds, must be of a certain amount, neither more nor less; and the business of mathematicians is to find this necessary amount, or, conversely, having given the amount, to find the curve in which it makes the body move. The connexion between the inquiries which form the main subject of the two first books of the Principia and physical astronomy, the subject of the third, is thus explicitly stated; but a plain indication is also here afforded of the great discovery in which the whole investigation is to end.

The doctrines of dynamics, known previously to his discoveries, are then given in the form of corollaries to the three general laws of motion. The first law is that of the *vis inertiæ,* already explained; and it is to be observed here that a steady and clear conception of the tendency of all moving bodies to proceed in a straight line unless deflected from it, is, perhaps, more than anything else, that which distinguished the Newtonian from the immediately preceding doctrines, mixing up as these did more influences than one proceeding from the centre with a view to explain the composite motion of the planets.

The second law is, that all changes in the motion of any body, or all changes from rest to motion, are in proportion to the moving force impressed, and are in the straight line of that force's direction.

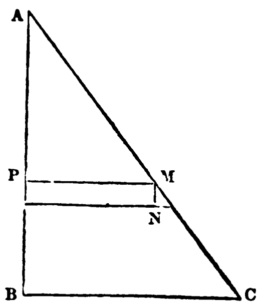
The third law is, that reaction is always equal and opposite to action; or that the mutual actions of any two bodies are always equal to one another, and in opposite directions.

From these laws the six corollaries which are added deduce the fundamental principles of dynamics; and there is a scholium to the whole, which states the application of those principles to the descent of heavy bodies and the parabolic motion of projectiles. Of all the principles, the most important is that of the Composition and Resolution of forces. As by the first law a body always perseveres in the straight line it moves in, unless in so far as some other force alters its direction; and as by the said law any new force impressed tends to move it in its own direction, it follows that, if two forces, not in the same or in directly opposite directions, act at one time, and by an instantaneous impulse, on any body, it must move in such a direction as that it shall be found both in a line parallel to the direction of the one force, and in a line parallel to the direction of the other; that is to say, in the diagonal of a parallelogram whose two contiguous sides are in the directions of the two forces, and are respectively equal to the space each force would carry it through in its own direction. Moreover, as each force separately would have carried it to the end of the line of its direction in the given time, it must move through the diagonal in the same time which it would have taken to move through either side if either force had acted alone. Thus the direction of every motion occasioned by any two forces acting at an angle to each other, may always be found by completing the parallelogram of which the directions of their forces are the contiguous sides; and so of any motion occasioned by any number of forces whatever acting angularly: and, conversely, every motion of a moving body may be resolved into two, of which the one is in any given direction whatever, and the other is found by completing the parallelogram, whereof that given direction is one of the sides and the direction the body moves in the diagonal. From this resolution of forces it is easily shown, that if any weights or other powers acting in parallel lines are applied to the opposite ends of a lever moving on a centre or fulcrum, the effect of each will be directly as its distance from that centre, in other words, as the length of the contiguous arm of the lever; consequently, that if the weights or powers are made inversely as those lengths, the whole will be in equilibrio or balanced. This is the well known and fundamental principle of the lever, the foundation of mechanics; and it applies also to the wheel and axle and the pully. The fundamental properties of the screw, the wedge, and the inclined plane are deduced in like manner from this important proposition. So may all the properties of the centre of gravity, and the method of finding it; for, in fact, the fulcrum of the lever is the common centre of gravity of two bodies equal to the two weights, and placed at the opposite ends of the lever, and the line joining the bodies is divided in the inverse proportion of those bodies. It also is easily shown that the common centre of gravity of two or more bodies is not moved, nor in any way affected, by their mutual actions on each other, but it either remains at rest, or moves forward in a straight line. So are the relative motions of any system of bodies, whether the space they occupy is at rest, or moves uniformly in a straight line.

The Scholium to the laws of motion first considers very briefly the motion of falling bodies which descend with a velocity uniformly accelerated, that velocity which is given to them by the attraction of the earth during the first instant continuing and having at each succeeding instant a new impulse added; the acceleration, therefore, is as the time; and they move through a space proportional to the velocity and the time jointly, consequently proportional to the square of the time, since the velocity is itself proportional to the time.* The scholium next,

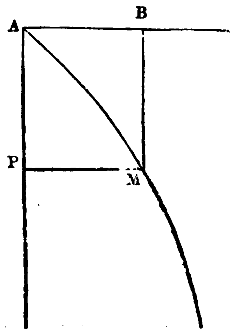
* Velocity is as time, i. e., v is as $m t$; space is as velocity × time; or s as $v \times t$, therefore space is as time × time, or as square of time, that is, s is as $m\ t \times t$, or $m\ t^2$.—The proportion of the space fallen through by the force of gravity (or moved through by any body uniformly accelerated) to the square of the times, is also demonstrated thus. Let the velocity acquired at any moment P of the time A P be P M, and because the velocity uni-

with equal brevity, states the projectile motion of heavy bodies. If a body be impelled in one direction by a force producing a uniform motion, and in another direction at any angle with the former by a force not uniform but accelerated, the diagonals which it will move through will at every instant change their direction towards the quarter to which the accelerating



formly increases, or as the time, P M : A P : : B C : A B, and therefore the line A C is a straight line, and the triangles A P M, A B C, are similar. But if q N is infinitely near P M, or P q represents the smallest conceivable time, the motion during that time may be conceived to be uniform and not accelerated. Now the space through which any body moves is as the velocity multiplied by the time ($s = v\,t$), therefore the space moved through in the time P q is as P q × q N. So the space moved through in the time A B will be as the sum of all the small rectangles P q × N q, or as the triangle A B C. But the triangle A B C is to any other of the triangles A P M as A B^2 : A P^2; therefore the spaces are as the squares of the times. he great general importance of this proposition which Galileo first proved, makes it necessary to have the demonstration clearly fixed in the reader's recollection.

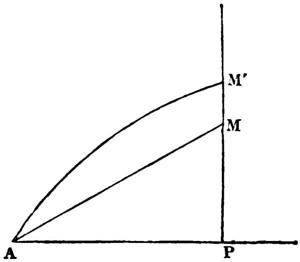
force tends. But a series of such diagonals is a polygon of an infinite number of sides, infinitely small; in other words, a curve line. Now in the case of a projectile, this continued or accelerating force is such as to make the body, if no other force acted on it, fall through spaces proportional to the square of the times. The other force acting once for all would make it, were there no gravity acting, move in spaces proportioned to the times simply. The latter or projecting force would make it move



through A B uniformly, or in spaces proportional to the times; the force of gravity would make it move through A P with a motion proportioned to the square of the times; therefore it will move in a curve passing through M, if P M is equal, and parallel to A B; and A P will be as the square of A B or P M, which is the property of the conic parabola $m . AP = PM^2$, m being the parameter to the point A. The scholium concludes by stating some consequences of the equality of action and reaction, the third law of motion, with respect to oscillation and impact, and also with respect to mutual attractions, of which the most important is that the attraction or weight of heavy bodies in respect of the earth, and of the earth in respect of them, is equal.

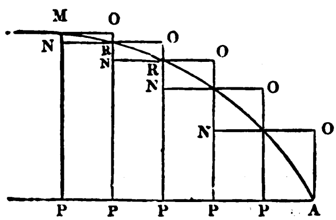
The great work itself, after these preliminary though essential matters, proceeds to its proper subject; but in order to show how the demonstrations are conducted, it prefixes a short treatise upon the method of Prime and ultimate Ratios, in eleven Lemmas, with their corollaries.

This method consists in considering all quantities as generated by the uniform progression or motion of other quantities, and examining the relations which the smallest conceivable spaces thus generated by this motion bear to one another, and to the spaces generated at the moment of their inception, or when they are nascent, which is termed their *prime ratio*, and at the moment of their vanishing, or when they are evanescent, which is termed their *ultimate ratio*. Thus a point moving along in a straightforward direction generates a straight line; a line moving parallel to itself, or two lines moving at right angles to one another, generate a rectangle: one line moving, while a point in it moves along it so that its progress on the moving line always bears a given ratio to the progress the line has made ($m.\,A\,P = P\,M$), describes a triangle; the same motion, if the pro-



gress of the point bears a variable relation to that of the line ($x.\,A\,P = P\,M'$; $x. \times A\,P$ being some function of A P), describes a curve line and curvilinear area; and so of solids, which are generated by the motion of planes.

It follows from this mode of generation that if the length of any curve line be divided into an infinite number of lines, the sum of these will not differ from the curve line by any assignable difference, nor will each differ from a straight line; and if its area be divided into an infinite number of smaller areas by lines drawn parallel to the line whose progressive motion generated the curvilinear area, the sum of these infinitely narrow areas will differ from the area of the curve by a difference less than any assignable difference, nor will each differ from a rectangle; in other words, the ratio of the nascent curve line and nascent curvilinear area will be that of equality with the small lines and small rectangles, and the ultimate ratio of the sums of the lines and rectangles to the whole curve line and curvilinear area, respectively, will be that of equality:—Or to put it otherwise, if the axis of the curve be divided into parts P P, &c., and the area into spaces P M R P, &c., by ordinates P M, P R, &c., and the number of these spaces be increased, and their breadth P P be diminished indefinitely,

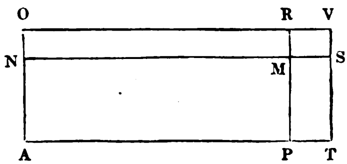


which is the operation of the generative motion of P M, the size of each of the small spaces M N R O by which the curvilinear areas differ from the rectangles diminishes infinitely, and the ultimate ratio of all the curve areas P M R P, and all the rectangles P N R P, becomes that of equality, and therefore the sum of evanescent differences N M O R, N R O R, &c., whereby the whole curvilinear area differs from the whole amount of the rectangles P N R P, becomes less than any assignable quantity, or the curvilinear area coincides with the sum of the rectangles. And so of the sum of all the diagonals M R, R R, &c., which becomes the curve line M R A.

Hence we infer that the amount of these small spaces or quantities N M O R, formed by multiplying together two evanescent quantities, is as nothing in comparison with the rectangles P M O P formed by only one evanescent quantity multiplied into a finite quantity, and may be neglected in any equation that expresses the relations of those rectangles with each other. But if some other quantities be found which are, in comparison with these small ones, themselves infinitely small, the areas formed by multiplying this second set of small quantities may be rejected in any equation expressing the relations of those first small quantities. Thus we have the origin and constitution of quantities which in the Newtonian scheme are called *fluxions* of different orders, because conceived to express the manner of the generation of quantities by the motion of others, and in Leibnitz's language are called infinitesimals or *differences,* because conceived to express the constant addition of one indefinitely small quantity to another. Obtaining the fluxion, or the differences, from the quantity generated by the motion or by the addition, is called the *direct method;* obtaining the quantity generated from the fluxions, or finding the sum of all the differences, is called the *indirect method.* The one theory calls the direct method that of finding *fluxions,* the indirect that of finding *fluents;* the other theory calls the former *differentiation,* or finding *differentials,* the latter *integration,* or finding *integrals.* The two systems, therefore, in no one respect whatever differ except in their origin and language; their rules, principles, applications, and results, are the same. A different symbol has been used in the two systems; Newton expressing a fluxion by a point or dot, and the fluxion of that fluxion, or a second fluxion, by two dots, and so on. Leibnitz prefixes the letter *d,* and its powers d^2, d^3, &c., instead, to express the differentials. In like manner $\int$ for sum is used by the latter to express the integral, and *f* by the former for the fluent. Although the continental method of notation is now generally used, and is on the whole most convenient, yet it has its inconveniency, as the d is sometimes confounded with co-efficients of the variable quantities; it is in some respects too, not very consistent with itself; as by making $d\,x^2$ mean the square of the fluxion, or differential of x; whereas it, strictly speaking, appears to denote the differential of x^2. There can be no doubt, however, which notation is the most convenient in the extension of the system to the calculus of variations, where the symbol is δ; for although the variation of a fluxion may perhaps even more conveniently be expressed by $\delta\,\dot{x}$ than by $\delta\,d\,x$, yet the fluxion of a variation can with no convenience be expressed by $\dot{\overline{\delta\,x}}$, or otherwise than by $d\,\delta\,x$. The expression of second fluxions undeveloped is also far less convenient by the Newtonian notation. Thus the fluxion of $\frac{d\,y}{d\,x}$ is sometimes required to be expressed without developement, as in the expression for the radius of curvature, where it is often expedient not to develope it in the general equation, but to find $\frac{d\,y}{d\,x}$ in terms of x or y before taking its fluxion; yet nothing can be more clumsy than to place a dot over the fraction, whereas $d\left(\frac{d\,y}{d\,x}\right)$ is perfectly convenient.

Several important considerations arise out of the nature and origin of these infinitesimal quantities as we have described them, and to these considerations we must now shortly advert, as they give the rules for finding the fluxions of all quantities, and, conversely, lead to those for investigating or finding the fluents of fluxional expressions.

A rectangle A M being generated by the side P M moving along A P while the side N M moves along A N, the movement or fluxion of A M, or of $AP \times PM$, is $PS + MO$, part of the gnomon



$PS + SO$, because the rectangle M V is evanescent compared with the other two, and is to be rejected. Therefore the fluxion of $AP \times PM = PM \times PT + NM \times NO$, or $PM \times PT + AP \times MR$. Calling $AP = x$, and $PM = y$, and $PT = dx$, and $MR = dy$, we have the fluxion of $xy = xdy + ydx$. But if the figure be a square, and $AP = PM$, or $x = y$, then the fluxion is $2xdx$. So if we would find the fluxion of a parallelopiped whose sides are x, y, and z, we shall in like manner find that it is $xydz + xzdy + yzdx$; if $x = z$, then it is

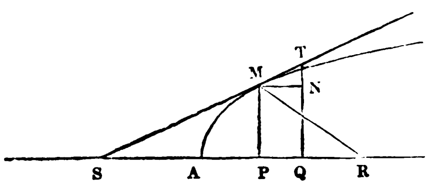
$2yx\,dx+x^2\,dy$; and if $x=y=z$, or the figure be a cube, it is $3x^2\,dx$. From hence, although the geometrical analogy serves us no further (as there are only three dimensions in figures), we derive by analogy the rule that the fluxion of x^m is $m\,x^{m-1}\,d\,x$. Also there is no dimension of figure less than unity; but by the same analogy we obtain the fluxion of x^{-m}, or $\frac{1}{x^m}$, namely, $m\,x^{-m-1}\,d\,x$, or $-\frac{m\,d\,x}{x^{m+1}}$, and of $\frac{x^m}{y^n}$, or $x^m \times y^{-n} = m\,x^{m-1}\,y^{-n}\,d\,x - n\,x^m\,y^{-n-1}\,d\,y$, or

$$= \frac{m\,x^{m-1}\,y^n\,d\,x - n\,x^m\,y^{n-1}\,d\,y}{y^{2n}}.$$

Consistently with the same principles, we may deduce this rule otherwise and more strictly. Let $x + d\,x$ be the quantity when increased by the fluxion. This multiplied by itself, or its square when completed, is $x^2 + 2\,x\,d\,x + (d\,x)^2$; but to have the mere increment or fluxion we must deduct x^2, and we must also reject $(d\,x)^2$ as evanescent compared with the function $2\,x\,d\,x$, which leaves $2\,x\,d\,x$ for the fluxion. So the cube is $x^3+3\,x^2\,d\,x + 3\,x\,(d\,x^2)+(d\,x)^3$, and rejecting, in like manner, we have $3\,x^2\,d\,x$; and by the binomial theorem $(x+d\,x)^m$ is $x^m + m\,x^{m-1}\,d\,x$, + &c. + $(d\,x)^m$, of which only the second term can upon the same principles be retained; that is $m\,x^{m-1}\,d\,x$; and the same rules apply to the fluxions of surds; so that the fluxion of $(x+y)^{\frac{1}{2}}$ is $\dfrac{d\,x+d\,y}{2\sqrt{x+y}}$.

It also follows that the fluent is a quantity such that, by taking its increment or fluxion according to the foregoing principles, you obtain the given fluxional expression. Thus if we have to integrate any quantity as $x^m\,d\,x$, we divide by $m+1$, and increase the exponent by unity, and erase the fluxional quantity. So that $\dfrac{x^{m+1}}{m+1}$ is the fluent required. But as every multiplication of any two quantities whatever gives a finite product, and every involution a finite power, while we can only divide so as to obtain a finite quotient, or extract so as to obtain a finite root, where the dividend or the power operated upon happen to be a perfect product or a perfect power; so in like manner we can only obtain the exact fluent or integral where the expression submitted to us is a complete fluxion. Thus, though such an expression as $\dfrac{x\,d\,x}{\sqrt{1+x^2}}$ is integrable, such an expression as $\dfrac{d\,x}{\sqrt{1+x^2}}$ is not integrable, for want of the x in the numerator, and various approximations and other contrivances are resorted to in order to accomplish or, at least, approach this object, of which the methods of series, of logarithms, and circular arcs are the most frequently used. The simplest case of integration by series may be understood in examples like the last; for if the square root be extracted by a series, we may be able to integrate each term, and so by the sum of the integrals to approach the real value of the whole.

From the doctrine as now explained, and the original foundations of the method as traced above, it follows that a variety of the most important problems may be solved with ease and certainty, which by the ancient geometry could only in certain cases, or by a happy accident, be investigated. Thus the tangents of curves may be found. For as the subtangent $SP : PM :: MN : TN$, $SP = \frac{PM \times MN}{NT}$



$= \frac{y\,dx}{dy}$, and so the perpendicular may always be drawn, for the subnormal $RP = \frac{PM^2}{SP} = \frac{y^2\,dy}{y\,dx}$

$= \frac{y\,d\,y}{d\,x}$. Therefore we have only to insert the one of these quantities in terms of the other from the equation between x and y (the equation to the curve), and we get the expression for the subtangent subnormal. Thus in the common parabola, whose equation is $y^2 = a\,x$, $\frac{y\,d\,x}{d\,y} = \frac{2\,y\,d\,y}{a} \times \frac{y}{d\,y} = \frac{2\,y^2}{a}$ or $2\,x$, and in the hyperbola, whose equation is $x\,y = a^2$, it (the subtangent) is $-x$. So in the circle $\frac{y\,d\,y}{d\,x}$ (the subnormal) $= r - x$ (r being the radius); all which we know from geometrical demonstration to be true.

Next, it is evident that when a quantity increasing has attained its maximum, it can have no further increment; or when decreasing it has attained its minimum, it can have no further decrement; consequently in such cases the fluxion of the quantity is equal to nothing. Hence a ready solution is afforded of the problems of maxima and minima. Thus would we know the proportion which two sides of a rectangle must have to each other, in order that, their sum being given, they may form a rectangle containing the greatest space possible; the fluxion of the rectangle must be put equal to nothing. Thus their sum being $= a$, the quantities are x and $a-x$, and their rectangle is $a x-x^2$, its fluxion $a\,d x-2\,x\,d x$, and this being put $=0$, we have $a\,d x=2\,x\,d x$, or $x=\frac{a}{2}$; therefore the figure must be a square. So would we know the point of the parabola $(b-x)^2=a\,(y-c)$ where the curve comes nearest the line b, the ordinate y must be a minimum, and $d y=0$. Now $y=\frac{(b-x)^2}{a}+c$, and $d y=\frac{2\,(b-x)}{a}\times-d x$, which being put $=0$ gives us $x=b$; or, at the extremity of the line b, the curve approaches the nearest; and that whatever be its parameter; for a has vanished from the equation.

Again, we have seen that the ultimate ratio of the sum of all the rectangles M P, P Q, contained by the ordinates and the increments of the abscissa to the curve's area A P M is that of equality; or, in other words, that the fluxion of a curvilinear area is the rectangle contained by the ordinate and the fluxion of the abscissa, or $y\,d x$, the fluent of this, or the sum of all those small rectangles being equal to the area. In this expression then, let y be inserted in terms of x, and the integral gives the area. Thus in the parabola $y=\sqrt{a}\,x$, therefore $d x\,\sqrt{a}\,x$ is the fluxion of the area and its fluent, or which is the same thing, the fluent of $\frac{2\,y^2\,d\,y}{a}$, is $\frac{2}{3} \times \frac{y^3}{a}$, or $\frac{2}{3} \times \frac{y^2}{a} \times y$, that is, $\frac{2}{3}\,x\,y$, or two-thirds of the rectangle of the co-ordinates, as we know from conic sections.

Next, we have seen that the ratio of the infinitely small rectilinear sides into which a curve line may be divided (each of those small lines being the hypothenuse of a right angle triangle, the sides of which are the fluxions of the co-ordinates N T, or M N), to the infinitely small portions of the curve itself is that of equality; therefore the fluxion of the curve is equal to the square root of the sum of the squares of the fluxions of the ordinate and abscissa, and that fluxion is equal to $\sqrt{d\,x^2 + d\,y^2}$. Hence in the circle, an arc whose cosine is x and radius r is equal to the fluent of $\frac{r\,d\,x}{\sqrt{r^2 - x^2}}$. And an arc whose cosine is $r - x$, is equal to the fluent of $\frac{r\,d\,x}{\sqrt{2\,r\,x - x^2}}$.

Again, because solids may in like manner be considered as composed of infinitely thin solids or plates, one placed upon the other, their fluxion is the area of the surface multiplied by the fluxion of the axis. Thus the base of any solid generated by the revolution of a surface rectilinear or curved must be a circle, and the proportion of the radius to the circumference being taken as $r : c$, y being the ordinate to the line bounding the vertical section, the surface will be $\frac{c\,y^2}{2\,r}$ and the fluxion of the axis x being $d\,x$, the fluxion of the solid will be $\frac{c\,y^2\,d\,x}{2\,r}$, in which y in terms of x being inserted from the boundary line's equation, the fluent gives the solid content. Thus if the line which bounds is straight and parallel to the axis, or the solid is a cylinder, its content is the circle multiplied by the axis; and if the line is drawn to a point in the axis, or the solid is a cone, then its content is one-third of the same product, or one-third of the cylinder, a well-known property of those two figures, proved by ordinary geometry. So in like manner we find the sphere to be two-thirds of the circumscribing cylinder, the celebrated discovery of Archimedes, of which he caused the diagram to be inscribed on his tomb.

Lastly, it may in like manner be shown that the radius of the osculating circle at any point of any curve, that is, the circle touching it at such point, and having the same curvature with it at that point, is equal to $\dfrac{(d\,x^2+d\,y^2)^{\frac{3}{2}}}{\pm\,d\,x^2 \times d\left(\frac{d\,y}{d\,x}\right)}$, where $d\,y$ being found in terms of x, the fluxion of $\frac{d\,y}{d\,x}$ is to be taken, so that there will in the result in each case be no fluxions at all. Thus in the parabola $y^2=2\,a\,x$, the radius of curvature is $=\sqrt{\dfrac{2\,x+a}{a}}\times(2x+a)$.

In all these operations, however, it must be observed, that as constant or invariable quantities have no fluxions, so when we reverse the operation and find fluents from given fluxional expressions, we never can tell whether a constant must not be added in order to complete that quantity, by taking whose fluxion the given expression was originally obtained. The determining of this constant quantity, and the finding whether there be any or not, depends upon the particular conditions of each problem. It is always added as a matter of course. Thus when we integrate $d\,x+d\,y$, we cannot tell whether this quantity arose from taking the fluxions of x and y only, or from taking the fluxion of $x+y+c$, and it must depend upon the nature of the question whether c

is to be added to the fluent or no; and if to be added, how it shall be ascertained.

Having explained this important method of investigation, by the help of which Newton was enabled to make his greatest mathematical discoveries, and by the principles of which he demonstrates them in the Principia, it only remains, before proceeding to the analysis of those discoveries, that we should remark the preference which he gives to the geometrical methods, improved and adapted to his purpose by the doctrine of prime and ultimate ratios. He uses this doctrine similar in principle to, and the foundation of, the noble and refined calculus which we have been considering; but he does not at all employ that calculus.

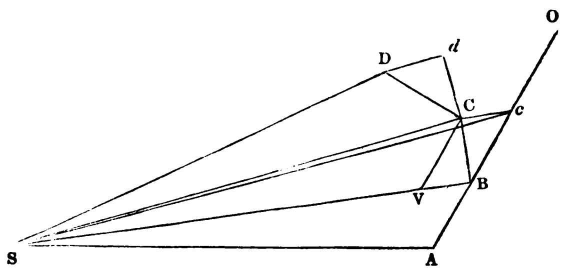
The First book treats of the motion of bodies without regard to the resistance of the medium that fills the space in which they move; and it is principally devoted to the consideration of motions in orbits determined by centripetal forces, and to examine the attraction of bodies. The Second book treats of the resistance of fluids chiefly as affecting the motions of bodies that move in them. The Third book contains the application of the principles thus established to the motions, attractions, and figures of the heavenly bodies.

I.

The fundamental proposition, as it may justly be termed, of the whole system, is one which Newton's predecessors may be said to have nearly reached; which Kepler, had he been more inclined to trust demonstration than empirical observation, certainly would have attained; and which Galileo would most certainly have discovered had he contemplated the facts discovered by Kepler, particularly his second law*—it is this. If any body is driven by any single impulse or force of projection, and is also drawn by another force so as to revolve round a fixed centre, the radius vector, or line drawn from the body to that centre, describes areas which are in the same fixed plane, and are always proportional to the times of the body's motion; and conversely, if any body which moves in any curve described in a plane so that the radius vector to a point either fixed or moving uniformly in a straight line, describes areas proportional to the times of the body's motion, that body is acted on by a centripetal force tending towards and drawing it to the point.

* See the historical notice above respecting this second law, viz., that the planets describe areas proportional to the times by their radii vectores.

To prove this, we have to consider that if a body moves equably on in a straight line, the areas or triangles which are described by a line drawn from it to any point are proportional to the portions of the straight line through which the body moves, that is, to the time, since it moves through equal spaces in equal times, because those triangles, having the same altitude, are to one another in the proportions of their bases. S being the point and A O the line of motion, S A B is to S B *c* as A B to B *c*.

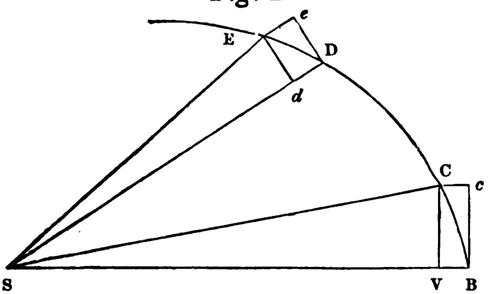


If then at B a force acts in the line S B, drawing the body towards S, it will move in the diagonal B C of a parallelogram of which the sides are B *c* and B V, the line through which the deflecting force would make it move if the motion caused by the other force ceased. C *c* therefore is parallel to V B, and the triangle S B C is equal to the triangle S B *c*; consequently the motion through A B and B C, or the times, are as the two triangles S A B and S B C:

and so it may be proved if the force acting towards S again deflects the body at C, making it move in the diagonal C D. If, now, instead of this deflecting force acting at intervals A, B, C, it acts at every instant, the intervals of time becoming less than any assignable time, and then the spaces A B, B C, C D will become also indefinitely small and numerous, and they will form a curve line; and the straight lines drawn from any part of that curve to S will describe curvilinear areas, as the body moves in the curve A B C D, those areas being proportional to the times. So conversely, if the triangles S B *c* and S B C are equal, they are between the same parallels, and *c* C is parallel to S B, and D *d* to S C; consequently the force which deflects acts in the lines S B and S C, or towards the point S. It is equally manifest that the direction of the lines B *c*, C *d*, from which the centripetal force deflects the body, is that of a tangent to the curve which the body describes, and that consequently the velocity of the body is in any given point proportional to the perpendicular drawn from the centre to the tangent; the areas of the triangles whose bases are equal, being in the proportion of their altitude, that is, of those perpendicular and those areas vary by the proportion, proportional to the times.

There are several other corollaries to this important proposition which deserve particular attention. In *fig.* 2, B *c* and D *e* are tangents to the curve at B and D respectively, B C and D E the arcs described in a given time; C *c* and E *e* lines parallel to the radii vectores S B and S D respectively,

Fig. 2.



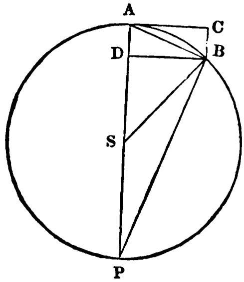
and C V, E *d* to the tangents. The centripetal forces at B and D must be in the proportion of V B and *d* D (being the other sides of the parallelogram of forces) if the arcs are evanescent, so as to coincide with the diagonals of the parallelograms V C and *d e*. Hence the centripetal forces in B and D are as the versed sines of the evanescent arcs; and the same holds true if instead of two arcs in the same curve, we take two arcs in different but similar curves.*

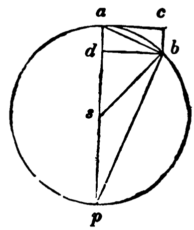
* If B C, D E, are bisected, the proportion is found with the halves of V B, D *d*; and that is the same proportion with the whole versed sines.

From these propositions another follows plainly, and its consequences are most extensive and important. If two or more bodies move in circular orbits (or trajectories) with an equable motion, they are retained in those paths by forces tending towards the centres of the circles, and those forces are in the direct proportion of the squares of the arcs described in a given time, and in the inverse proportion of the radii of the circles.

First of all it is plain, by the fundamental proposition, that the forces tend to the centres S, *s*, because the sectors A S B and P B S being as the arcs A B, B P, and the sectors *a s b*, *b s p*, as the arcs *a b*, *b p*, which arcs being all as the times, the areas are proportional to those times of describing them, and therefore S *c s* are the centres of the deflecting forces. Then, drawing the tangents A C, *a c*, and completing the parallelograms D C, *d c*, the diagonals of which coincide with the evanescent arcs A B, *a b*, we have the centripetal forces in A and *a*, as the versed sines A D, *ad*. But because A B P and *a b p* are right angles (by the property of the circle), the triangles A D B, A P B, and *a d b*, *a p b*, are respectively similar to one another. Wherefore A D : A B : : A B : A P and $AD = \frac{AB^2}{AP}$, and in like manner

$a\,d = \dfrac{a\,b^2}{a\,p}$, or as the evanescent arcs coincide with





the chords, $A\,D = \text{arc}\,\dfrac{A\,B^2}{A\,P}$ and $a\,d = \text{arc}\,\dfrac{a\,b^2}{a\,p}$. Now these are the properties of any arcs described in equal times; and the diameters are in the proportion of the radii; therefore the centripetal forces are directly as the squares of the arcs, and inversely as the radii.

It is difficult to imagine a proposition more fruitful in consequences than this; and therefore it has been demonstrated with adequate fulness. In the *first* place, the arcs described being as the velocities, if F, f are the centripetal forces, and V, v the velocities, and R, r the radii, $F : f :: V^2 : v^2$; and also $:: r : R$, or $F : f :: \dfrac{V^2}{R} : \dfrac{v^2}{r}$. Now as in the circle V and R, v and r are both constant quantities, the centripetal force is itself constant, which retains a body by deflecting it towards the centre of the circle.

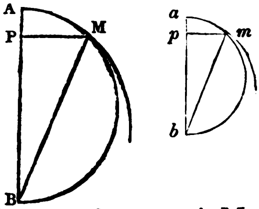
Secondly. The times in which the whole circles are described (called the periodic times) are as the total circumferences or peripheries; $T : t :: P : p$, but the peripheries are as the radii or $:: R : r$. Therefore $T : t :: R : r$; also $V : v :: \frac{P}{T} : \frac{p}{t}$, therefore inversely as the radii, or $T : t :: \frac{R}{V} : \frac{r}{v}$, and $V^2 : v^2 :: \frac{R^2}{T^2} : \frac{r^2}{t^2}$. But the centripetal forces $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$; substituting for the ratio of $V^2 : v^2$, its equal the ratio of $\frac{R^2}{T^2} : \frac{r^2}{t^2}$, $F : f :: \frac{R}{T^2} : \frac{r}{t^2}$; or the centripetal forces are directly as the distances and inversely as the squares of the periodic times; the forces being as the distances if the times are equal; and the times being equal if the forces are as the distances. It also follows that if the periodic times are as the distances, $F : f :: \frac{R}{R^2} : \frac{r}{r^2}$; that is, $:: \frac{1}{R} : \frac{1}{r}$ or inversely as the distances. In like manner if the periodic times are in proportion to any power n, of the distance, or $T : t :: R^n : r^n$, we shall have $T^2 : t^2 :: R^{2n} : r^{2n}$ and $F : f ::$

$\frac{R}{R^{2n}} : \frac{r}{r^{2n}}$; that is :: $\frac{1}{R^{2n-1}} : \frac{1}{r^{2n-1}}$; and conversely if the centripetal force is in the inverse ratio of the $(2n-1)^{th}$ power of the distance, the periodic time is as the n^{th} power of that distance. Likewise as the velocities of the bodies in their orbits or $V : v :: \frac{R}{T} : \frac{r}{t}$, if we make $T : t :: R^n : r^n$, then $V : v :: \frac{R}{R^n} : \frac{r}{r^n}$, or :: $\frac{1}{R^{n-1}} : \frac{1}{r^{n-1}}$. Thus, suppose n is equal to $\frac{3}{2}$ we have for the velocities $V : v :: \frac{1}{\sqrt{R}} : \frac{1}{\sqrt{r}}$, or they are in the inverse subduplicate proportion of the distances; and for the centripetal forces we have $F : f :: \frac{1}{R^{3-1}} : \frac{1}{r^{3-1}} :: \frac{1}{R^2} : \frac{1}{r^2}$ or the attraction to the centre is inversely as the square of the distance. Now if $n = \frac{3}{2}$, $T : t :: R^{\frac{3}{2}} : r^{\frac{3}{2}}$ or $T^2 : t^2 :: R^3 : r^3$; in other words the squares of the periodic times are as the cubes of the distances from the centre, which is the law discovered by Kepler actually to prevail in the case of the planets. And as he also showed that they describe equal areas in equal times by their radii vectores drawn to the sun, it follows from the fundamental proposition, *first*, that they are deflected from the tangents of their orbits by a power tending towards the sun; and then follows, *secondly*, from this last deduction respecting it, the proportion of $F : f :: \frac{1}{R^2} : \frac{1}{r^2}$, that this central force acts inversely as the squares of the distances, always supposing the bodies to move in circular orbits, to which our demonstration has hitherto been confined.*

The extension, however, of the same important proposition to the motion of bodies in other curves is easily made, that is to the motion of bodies in different parts of the same curve or of curves which are similar. For in evanescent portions of the same curve, the osculating circle or circle which has the same curvature at any point coincides with the curve at that point; and if a line is drawn to the extremity of that circle's diameter, A M B and *a m b* may be considered as triangles; and as they are right angled at M and *m*, AM^2 is equal to $AP \times AB$ and am^2 to $ap \times ab$; and where the curvature is the same as in corresponding points of similar curves, those

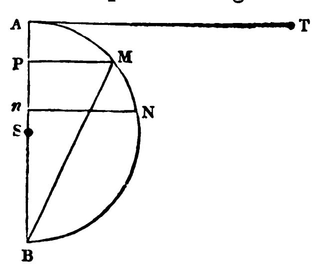
* We shall afterwards show from other considerations, that this sesquiplicate proportion only holds true on the supposition of the bodies all moving without exerting any action on each other, when we come to consider Laplace's theorems on elliptical motion.

squares are proportional to the lines A P, or *a p*, or



those versed sines of the arcs A M and *a m* are proportional to the squares of the small arcs. Hence if the distances of two bodies from their respective centres of force be D, *d*, the deflecting force in any points A and *a* being as the versed sines, those forces are as A M^2 : $a\ m^2$; and from hence follows generally in all curves, that which has been demonstrated respecting motion in circular orbits. The planets then and their satellites being known by Kepler's laws to move in elliptical orbits, and to describe round the sun in one focus areas proportional to the times by their radii vectores drawn to that focus, and it being further found by those laws that the squares of their periodic times are as the cubes of the mean distances from the focus, they are by these propositions of Sir Isaac Newton which we have been considering, shown to be deflected from the tangent of their orbit, and retained in their paths by a force acting inversely as the squares of the distances from the centre of motion.

But another important corollary is also derived from the same proposition. If the projectile or tangential force in the direction A T ceases, the body instead of moving in any arc A N, is drawn by the same centripetal force in the straight line A S. Let A *n* be the part of A S, through which the body falls by the force of gravity, in the same time that it would take to describe the arc A N. Let A M be the infinitely small arc described in an instant; and A P its versed sine. It was before shown, in the corollaries to the first proposition, that the centripetal force in A is as A P, and the body would move by that force through A P, in the same time in which it describes the arc A M. Now the force of gravity being one which operates like the centripetal force at every instant, and uniformly accelerates the descending body, the spaces fallen through will be as the squares of the times. Therefore, if A *n* is the space through which the body



falls in the same time that it describes A N, A P is to A n as the square of the time taken to describe A M to the square of the time of describing A N, or as $AM^2 : AN^2$, the motion being uniform in the circular arc. But A M, the nascent arc, is equal to its chord, and A M B being a right angled triangle as well as A P M, $AB : AM :: AM : AP$ and $AP = \frac{AM^2}{AB}$. Substituting this in the former proportion, we have $\frac{AM^2}{AB} : An :: AM^2 : AN^2$, or $An : AN^2 :: \frac{AM^2}{AB} : AM^2$, that is $:: 1 : AB$. Therefore $AN^2 = An \times AB$, or the arc described, is a mean proportional between the diameter of the orbit, and the space through which the body would fall by gravity alone, in the same time in which it describes the arc.

Now let A M N B represent the orbit of the moon; A N the arc described by her in a minute. Her whole periodic time is found to be 27 days 7 hours and 43 minutes, or 39,343 minutes; consequently $AN : 2ANB :: 1 : 39{,}343$.

Now the mean distance of the moon from the earth is about 30 diameters of the earth, and the diameter of her orbit, 60 of those diameters; and a great circle of the earth being about 131,630,572 feet, the circumference of the moon's orbit must be 60 times that length, or 7,897,834,320, which being divided by 39,343 (the number of minutes in her periodic time), gives for the arc A N described in one minute 200,743, of which the square is 40,297,752,049 ($A N^2$), which (by the proposition last demonstrated) being divided by the diameter A B gives A n. But the diameter being to the orbit as 1 : 3.14159 nearly, it is equal to about 2,513,960,866. Therefore A n = 16.02958, or 16 feet, and about the third of an inch. But the force which deflects the moon from the tangent of her orbit, has been shown to act inversely as the square of the distance; therefore she would move 60 × 60 times the same space in a minute at the surface of the earth. But if she moved through so much in a minute, she would in a second move through so much less in the proportion of the squares of those two times, as has been before shown. Wherefore she would in a second move through a space equal to $16\frac{1}{34}$ nearly (16.02958). But it is found by experiments frequently made, and among others by that of the pendulum,* that a body falls about this space in

* It is found that a pendulum vibrating seconds, is about the

one second upon the surface of the earth. Therefore the force which deflects the moon from the tangent of her orbit, is of the same amount, and acts in the same direction, and follows the same proportions to the time that gravity does. But if the moon is drawn by any other force, she must also be drawn by gravity; and as that other force makes her move towards the earth 16 feet $\frac{1}{3}$ inch, and gravity would make her move as much, her motion would therefore be 32 feet $\frac{2}{3}$ inch in a second at the earth's surface, or as much in a minute in her orbit; and her velocity in her orbit would therefore be double of what it is, or the lunar month would be less than 13 days and 16 hours. It is, therefore, impossible that she can be drawn by any other force, except her gravity, towards the earth.*

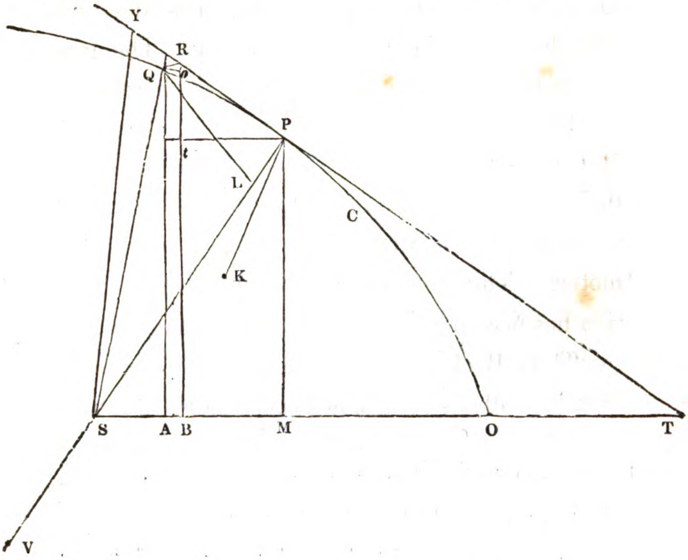
Such is the important conclusion to which we

length of 3 feet $3\frac{1}{3}$ inches in this latitude; and the space through which a body falls in a second is to half this length as the square of the circumference of a circle to that of the diameter, or as 9.8695 : 1, and that is the proportion of the half of 3 feet $3\frac{1}{3}$ inches to somewhat more than 16 feet.

* The proposition may be demonstrated by means of the Prop. XXXVI. of Book I., as well as by means of the proposition of which we have now been tracing the consequences (Prop. IV). But in truth the latter theorem gives a construction of the former problem (Prop. XXXVI.), and from it may be deduced both that and Prop. XXXV.

are led from this proposition, that the centripetal forces are as the squares of the arcs described directly, and as the distances inversely. The great discovery of the law of the universe, therefore, is unfolded in the very beginning of the Principia. But the rest of the work is occupied with tracing the various consequences of that law, and first of all in treating generally of the laws of curvilinear motion. The demonstration of the moon's deflection has been now anticipated and expounded from the Third Book, where it is treated with even more than the author's accustomed conciseness. But there seemed good ground for this anticipation, inasmuch as the Scholium to the Fourth Proposition refers in general terms to the connexion between its corollaries, and the Theory of Gravitation.

The versed sine of the half of any evanescent arc (or sagitta of the arc) of a curve in which a body revolves, was proved to be as the centripetal force, and as the square of the times; or as $F \times T^2$. Therefore the force F is directly as the versed sine, and inversely as the square of the time. From this it follows that the central force may be measured in several ways. The arc being Q C, we are to measure the central force in its middle point P. Then the areas being as the times; twice the triangle S P Q, or Q L ×



S P is as T in the last expression; and, therefore, Q R being parallel to L P, the central force at P is as $\frac{QR}{SP^2 \times LQ^2}$. So if S Y be the perpendicular upon the tangent P Y, because P R and the arc P Q, evanescent, coincide, twice the triangle S P Q is equal to S Y × Q P; and the central force in P is as $\frac{QR}{SY^2 \times QP^2}$. Lastly, if the revolution be in a circle, or in a curve having at P the same curvature with a circle whose chord passes from that point through S to V, then the measure of the central force will be $\frac{1}{SY^2 \times PV}$. By finding the value of those solids in any given curve, we can determine the centripetal force in terms of the radius vector S P; that is, we can find the proportion which the force must bear to the distance, in order to retain the body in the given orbit or trajectory; and conversely, the force being given, we can determine the trajectory's form.

This proposition then, with its corollaries, is the foundation of all the doctrine of centripetal forces, whether direct or inverse, that is, whether we regard the method of finding, from the given orbit, the force and its proportion to the distance, or the method of finding the orbit from the given force. We must, therefore, state it more in detail, and in the analytical manner, Sir Isaac Newton having delivered it synthetically, geometrically, and with the utmost brevity.

It may be reduced to five kinds of formulæ

1. If the central force in two similar orbits be called F and f, the times T and t, the versed sines of half the arcs S and s, then $F : f :: \frac{S}{T^2} : \frac{s}{t^2}$ and generally F is as $\frac{2\,S}{T}$.

2. But draw S P to any given point of the

orbit in the middle of an infinitely small arc Q C. Let T P touch the curve in P, draw the perpendicular S Y from the centre of forces S to P T produced, draw S Q infinitely near S P, and Q R parallel to S P, Q O and R O parallel to the co-ordinates S M, M P. Then P being the middle of the arc, twice the triangle S P Q is proportional to the time in which C Q is described. Therefore Q P $\times$ P S or Q L $\times$ P S is proportional to the time; and Q R is the versed sine of $\frac{CQ}{2}$, therefore F as $\frac{S}{T^2}$ becomes F as $\frac{QR}{LQ^2 \times SP^2}$; and if $SM = x$, $MP = Y$, and because the similar triangles Q R o and S M P give $QR = \frac{Qo \times SP}{SM}$, and because A M being the first fluxion of S M, O Q is its second fluxion (negatively), therefore Q R $=$ $\frac{-d^2x \times \sqrt{x^2+y^2}}{x}$ (taken with reference to $d\,t$ constant), and F is as $\frac{-d^2x\sqrt{x^2+y^2}}{x \times LQ^2 \times (x^2+y^2)}$. But $LQ^2 = QP^2 - LP^2$ and L P is the fluxion of S P or $\sqrt{x^2+y^2}$. Therefore $LQ^2 = \frac{(x\,dy+y\,dx)^2}{x^2+y^2} = \frac{y^4\left(d\frac{x}{y}\right)^2}{x^2+y^2}$ and F is as $\frac{-d^2x\sqrt{x^2+y^2}}{xy^4\left(d\frac{x}{y}\right)^2}$.

But as the fluxion of the time ($LQ \times PS$) may be made constant, QR will represent the centripetal force; and that force itself will therefore be as $-\frac{d^2 x \sqrt{x^2+y^2}}{x}$,* taken with reference to dt constant.

3. The rectangle $SY \times QP$ being equal to $QL \times SP$ and $SY = \frac{y\,dx - x\,dy}{\sqrt{dx^2 + dy^2}}$, we have F as

$$\frac{QR}{SY^2 \times QP^2} = \frac{QR}{(y\,dx - x\,dy)^2} = \frac{QR}{y^4\left(d\frac{x}{y}\right)^2}.$$

4. Because $F = \frac{QR}{SY^2 \times QP^2}$ and $\frac{QP^2}{QR}$ is equal to the chord PV of the circle, which has the same curvature with QPO in P, and whose centre is K (because $QP^2 = QR \times PV$ by the nature of the circle and the equality of the evanescent arc QP with its sine, and thus $PV = \frac{QP^2}{QR}$), therefore $\frac{QR}{QP^2} = \frac{1}{PV}$ and F is as $\frac{1}{SY^2 \times PV}$.

* Of these expressions, although I have sometimes found this, which was first given by Herrman, serviceable, I generally prefer the two, which are in truth one, given under the next heads. But the expression first given $-\frac{d^2 x \sqrt{x^2+y^2}}{x y^4 \left(d\frac{x}{y}\right)^2}$ is without integration an useful one.

In like manner if the velocity, which is inversely as S Y, be called v, F is as $\frac{v^2}{PV}$. Now the chord of the osculating circle is to twice the perpendicular S Y as the fluxion of S P to the fluxion of the perpendicular; and calling S P the radius vector r, and S Y p, we have $PV = \frac{2p\,dr}{dp}$, F is as $\frac{dp}{2p^3\,dr}$; and also F is as $\frac{v^2\,dp}{2\,dr}$. In these formulæ, substituting for p and r their values in terms of x and y, we obtain a mean of estimating the force as proportioned to r, which is $\sqrt{x^2 + y^2}$.

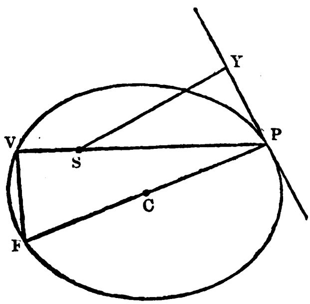
5. The last article affords, perhaps, the most obvious methods of arriving at central forces, both directly and inversely. Although the quantities become involved and embarrassing in the above general expressions for all curves, yet in any given curve the substitutions can more easily be made. A chief recommendation of these expressions is, that they involve no second fluxions, nor any but the first powers of any fluxions. But it may be proper to add other formulas which have been given, and one of which, at least, is more convenient than any of the rest.

One expression for the centrifugal force (and one sometimes erroneously given for the centripetal)* is $\frac{d s^2}{2 R}$, s being the length of the curve and R the radius of curvature; this gives a ready means of working if the radius is known. But its general expression involves second fluxions, the usual formula for it being $\frac{d s^3}{d x^2 \times d\left(\frac{d y}{d x}\right)}$; consequently we must first find $\frac{dy}{dx} = X$ (a function of x), and then there are only first fluxions.

Another for this radius of curvature is $\frac{d s^2}{\sqrt{(d^2 y)^2 + (d^2 x)^2}}$, and this is used by Laplace; and another is $\frac{r d r}{d p}$, which, with other valuable formulas, is to be obtained from Maclaurin's Fluxions. But the formula generally ascribed to John Bernouilli (Mém. Acad. des Sciences, 1710), is, perhaps, the most elegant of any, $F = \frac{r}{2. p^3 \times R}$; and this results from substituting 2 R for its value $\frac{2 r d r}{d p}$, in the equation to F, deduced above from Newton's formula, namely, $F = \frac{d p}{2 p^3 d r}$.

* This error appears to have arisen from taking the case where the radius of curvature and radius vector coincide, that is, the case of the circle, in which the centrifugal and centripetal forces are the same.

But the proposition is so important, that it may be well to prove it, and to show that it is almost in terms involved in the third corollary to Prop. VI. Book I. of the Principia. By that corollary $F = \frac{1}{p^2 . C}$ (C being the osculating circle's chord which passes through the centre of forces). But drawing S Y, the perpendicular to the tangent,



and P C F through the centre of the circle, and joining V F, which is, therefore, parallel to Y P, we have V P : P F :: S Y : S P or $C : 2\,R :: p : r$ and $C = \frac{2\,R . p}{r}$, which substituted for C in the above equation, gives $F = \frac{r}{2\,P^3 . R}$.

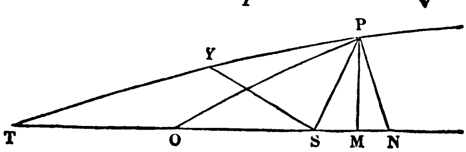
It is remarkable that the circumstance of this formula being thus involved in that of Sir Isaac Newton seems never to have been observed by Keill, who, in the Philosophical Transactions, xxvi. 74, gives a demonstration of it much more roundabout, and as of a theorem which Demoivre had communicated to him, adding, that Demoivre also informed him of Sir Isaac Newton having invented a similar method before. In fact, he had above 20 years before given it in substance, though not in express terms, in the Sixth Proposition, the addition of two lines to which at once would have led to this formula. But, again, when John Bernouilli, two years afterwards, wrote his letter to Herrman (Mém. Acad. des Sciences, 1710), he gives it as his own discovery, and as such it has generally been treated, with what reason we have just seen. He is at much pains to state, p. 529, that he had sent it in a letter to Demoivre in February, 1706; but the Principia had been published nineteen years before. Herrman, in his Phoronomia, erroneously considers the expression as discovered by Demoivre, Grandi, and Bernouilli. (Lib. I. Prop. XXII.)

In all these cases p is to be found first, and the expression for it (because, p. 301, TP : PM : : TS : SY and $TS = \frac{y\,dx - x\,dy}{dy}$, and $PT = \frac{y}{dy}\sqrt{dy^2 + dx^2}$)

is $p = SY = \dfrac{y\,dx - x\,dy}{\sqrt{dy^2 + dx^2}} = \dfrac{y^2\,d\dfrac{x}{y}}{\sqrt{dy^2 + dx^2}}$. Also

$r = \mathrm{S\,P} = \sqrt{x^2 + y^2}$. Then the radius of curvature $\mathrm{R} = \frac{(d\,x^2 + d\,y^2)^{\frac{3}{2}}}{d\,x^2 \times d\,\mathrm{X}}$ (X being $\frac{d\,y}{d\,x}$ in terms of x, and having no fluxion in it when the substitution for $d\,y$ is made.) Therefore, the expression for the centripetal force becomes $\frac{\sqrt{x^2 + y^2} \times d\,x^2 \times d\,\mathrm{X}}{2\,y^4\left(d\,\frac{x}{y}\right)^2}$, in which, when y and $d\,y$ are put in terms of x, as both numerator and denominator, will be multiplied by $d\,x^3$, there will be no fluxion, and the force may be found in terms of the radical—that is, of r, though often complicated with x also. It is generally advisable, having the equation of the curve, to find p, r, and R, first by some of the above formulæ, and then substitute those values, or $d\,p$ and $d\,r$, in either of the expressions for F, $\frac{d\,p}{2\,p^3\,d\,r}$ or $\frac{r}{2\,p^3\,\mathrm{R}}$.

To take an example in the parabola, where S being the focus, and $\mathrm{O\,S} = a$, $y^2 = 4\,a\,x$, and $\mathrm{T\,M} = 2\,x$, and $p = \mathrm{Y\,S} = \sqrt{(a + x)\,a}$; $r = \mathrm{S\,P} = a + x$, and $\mathrm{R} = \frac{r\,d\,r}{d\,p} = 2(a+x)$, $\sqrt{\frac{a+x}{a}}$;



we have therefore F as $\frac{r}{2p^3.R} =$

$$\frac{a+x}{2(a.a+x)^{\frac{3}{2}} \times 2(a+x)\sqrt{\frac{a+x}{a}}} = \frac{a+x}{4a(a+x)^3}$$

$\frac{1}{4a(a+x)^2} = \frac{1}{4.OS.SP^2}$, or, because $4.OS$ (the parameter) is constant, inversely as the square of the distance. and the other formula $F = \frac{dp}{2p^3dr}$ gives the same result $\frac{1}{4SP^2}$.*

Again, in the ellipse, if a be half the transverse axis, and b the eccentricity (or distance of the focus from the centre), and r the radius vector, we have $p = \sqrt{a^2 - b^2} = \sqrt{\frac{r}{2a-r}}$ and $dp = \frac{a\,dr}{\sqrt{r}(2a-r)^{\frac{3}{2}}}$, therefore the formula $\frac{dp}{2p^3.dr}$ becomes

$$\frac{a\,dr}{2\sqrt{r} \times (r)^{\frac{3}{2}} \times dr} = \frac{a}{2r^2},$$ or the force is inversely as the square of the distance.

Lastly, as the equations are the same for the hyperbola, with only the difference of the signs, the value of the force is also inversely as r^2, or the

* This result coincides with the synthetical solution of Sir Isaac Newton in Prop. XIII.

square of the distance. In the circle $b = 0$ and $a =$ the radius $= r = p$; hence the force is as $\frac{1}{2r}$, which being constant, the force is everywhere the same. But if the centre of forces is not that of the circle, but a point in the circumference, the force is as $\frac{1}{r^5}$.

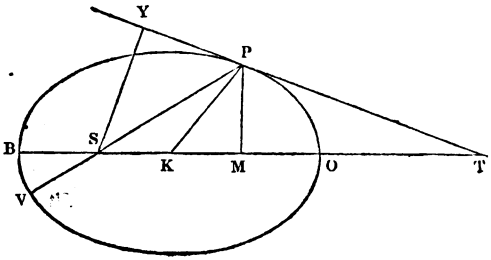
Respecting centrifugal forces it may be enough to add, that if v is the velocity and r the radius, the centrifugal force f, in a circle, is as $\frac{v^2}{r}$. Also if R be the radius of curvature, and f for any curve is $= \frac{v^2}{R}$. When a body moves in a circle by a centripetal force directed to the centre, the centrifugal force is equal and opposite to the centripetal. Also the velocity in uniform motion, like that in a circle, being as $\frac{s}{t}$, the space divided by the time, and the arc being as the radius r, f is as $\frac{v^2}{r.\ t^2}$ or as $\frac{r}{t^2}$. If two bodies moving in different circles have the same centrifugal force, then the times are as $\sqrt{r}$. It is to the justly celebrated Huygens that we owe the first investigation of centrifugal forces. The above propositions, except the second, are abridged from his treatise.*

The rest of the investigation of centripetal forces is an expansion of the formulas above given, and their application to various cases, but chiefly to the conic sections. It may be divided into four branches. *First*, the rules are given for determining the central force required to make the body move in a given orbit of one of the four conic sections. *Secondly*, the inquiry becomes material how curves of a given kind, namely, the conic sections, may severally be found by merely ascertaining certain points in them, or certain lines which they touch, because this enables us to ascertain, among other things, the whole of a planet's orbit, from ascertaining certain points by actual observation. This branch of the subject is purely mathematical, consisting of the rules for drawing those curves through given points, or between, or touching given straight lines; and it is subdivided into two heads according as one or neither focus is given. The *third* object is to ascertain the motion, place, and time of bodies moving in given trajectories generally; and, among others, also of bodies descending, or retarded in ascending, by gravity. The *fourth* branch treats

* Horologium Oscillatorium, Ed. 1673, p. 159, App.

of the converse inquiry into the figures of the trajectories, and the places, times, and motion, when the nature of the centripetal force is known.

It is thus manifest that the great importance of motion in the Conic Sections made Sir Isaac Newton consider those curves in particular, before discussing the general subject of trajectories.



i. In exemplifying the use of the formulas we have shown the proportion of the force to the distance in the conic sections generally, their foci being the centres of forces. Let us now see more in detail what the proportion is for the circle. If S is the centre of forces and K of the circle, P T a tangent, S Y a perpendicular to it, K M and M P co-ordinates, $SK = b$, $KO = a$, $PM = y$, and $MK = x$. Then, by similar triangles, T K P and T S Y, we have $SY =$

$\frac{ST \times KP}{TK}$ or (because the sub-tangent $MT = \frac{y^2}{x}$, and $a^2 = x^2 + y^2$) $\frac{a^2 + bx}{a}$ or $\left(\frac{2a^2 + 2bx}{2a}\right)$; also $SP = \sqrt{a^2 + 2bx + b^2}$, and because by the property of the circle $OS \times SB$ or $(a + b)(a - b) = a^2 - b^2 = PS \times SV$; therefore $SV = \frac{a^2 - b^2}{\sqrt{a^2 + 2bx + b^2}}$ and $PV = \frac{2a^2 + 2bx}{\sqrt{a^2 + 2bx + b^2}}$.

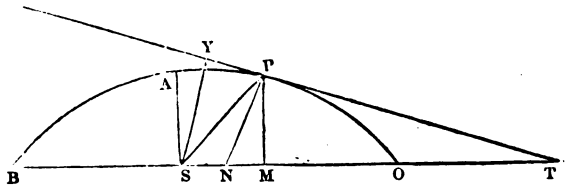
Now by the formula already stated as Bernouilli's, but really Sir Isaac Newton's, the centripetal force in P is as $\frac{SP}{2SY^3 \times R}$, R being the radius of curvature, and in the circle that is constant being $= a$, the semi-diameter; therefore the force is as $\frac{\sqrt{a^2 + 2bx + b^2}}{2a\frac{(2a^2 + 2bx)^3}{8a^3}}$

or as $\frac{4a^2 \times \sqrt{a^2 + 2bx + b^2}}{(2a^2 + 2bx)^3}$; that is $\frac{BO^2 \times SP}{(2a^2 + 2bx)^3}$,

or as $\frac{BO^2 \times SP^3}{(2a^2 + 2bx)^3 \times SP^2}$, or as $\dfrac{BO^2}{\dfrac{(2a^2 + 2bx)^3}{SP^3} \times SP^2}$. But $\dfrac{2a^2 + 2bx}{SP}$

$= \dfrac{2a^2 + 2bx}{\sqrt{a^2 + 2bx + b^2}} = PV$. Therefore the central force is as $\dfrac{BO^2}{PV^3 \times SP^2}$, or (because OB^2 is constant) the central force is inversely as the square of the distance and the cube of the chord jointly. Of consequence, where S is in the centre of the circle and $b = o$, the force is constant, the expression becoming $\dfrac{1}{2a^3}$; and if S is in the circumference of the circle as at B, or $a = b$, then the expression becoming $\dfrac{1}{\sqrt{2a} \times (a + x)^{\frac{5}{2}}}$, and the chord and radius vector coinciding, the force is inversely as the fifth power of the distance, and is also inversely as the $\frac{5}{2}$ power of the cosine of the angle PSO.

By a similar process it is shown that in an ellipse the force directed to the centre is as the distance. Indeed, a property of the ellipse renders this proof very easy. For if SY is the perpendicular to the tangent T P, and N P (the normal) parallel to S Y, and S A the conjugate axis; S A is a mean



proportional between S Y and P N, and therefore $SY = \frac{AS^2}{PN}$, also the radius of curvature of the ellipse is (like that of all conic sections) equal to $\frac{4PN^3}{P^2}$, P being the parameter. Therefore we have to substitute these values for S Y and the radius of curvature, R, in the expression for the central force, $\frac{SP}{2R \times SY^3}$ and we have $\frac{SP}{\frac{2 \times 4 . PN^3}{P^2} \times \frac{AS^6}{PN^3}}$

$= \frac{P^2}{8AS^6} \times SP$, therefore, neglecting the constant $\frac{P^2}{8AS^6}$, the centripetal force is as the distance directly.

From hence it follows, conversely, that if the centripetal force is as the distance, the orbit is elliptical or circular; for by reversing the steps of the last demonstration we arrive at an equation to the ellipse; or, in case of the two axes being equal, to the circle. It also follows that if bodies revolve in circular or elliptical orbits round the same centre, the centre of the figures being the centre of forces, and the force being as the distance, the periodic time of all the bodies will be the same, and the spaces through which they move, however differing from each other, will all be described in the same time. This proposition, which sometimes has appeared paradoxical to those who did not sufficiently reflect on the subject, is quite evident from considering that the force and velocity being increased in proportion to the distance, and the lengths of similar curvilinear and concentric figures being in some proportion, and that always the same, to the radii, the lengths are to each other as those radii, and consequently the velocity of the whole movement is increased in the same proportion with the space moved through. Hence the times taken for performing the whole motion must be the same. Thus, if V and v are the velocities, R and r the radii, S and s the lines described in the times T and t, by two such bodies round a common centre, $V : v :: R : r$, and $S : s :: R : r$; and because $V = \frac{S}{T}$ and $v = \frac{s}{t}, \frac{S}{T} : \frac{s}{t} :: R : r$, and $S : s :: T R : t r$; or $R : r :: T R : t r$; and therefore $T = t$. Hence if gravity were the same towards the sun that it is between the surface and centre of each planet, or if the sun were moved but a very little to one side, so as to be in the centre of the ellipse, the whole planets would revolve round him in the same time, and Saturn and Uranus would, like Mercury, complete their vast courses in about three of our lunar months instead of 30 and 80 years, —a velocity in the case of Uranus equal to 75,000 miles in a second, or nearly one-third that of light.

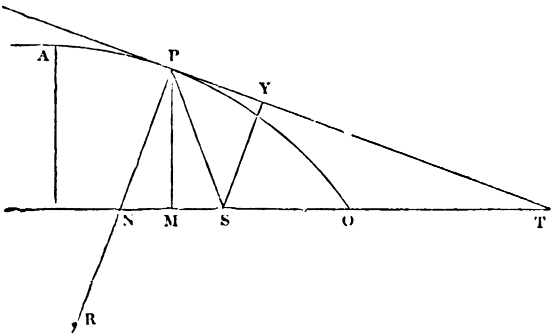
It also follows from this proposition that, if such a law of attraction prevailed, all bodies descending in a straight line to the centre would reach it in the same time from whatever distance they fell, because the elliptic orbit being indefinitely stretched out in length and narrowed till it became a straight line, bodies would move or vibrate in equal times through that line. This is the law of gravity at all points within the earth's surface, and Sir I. Newton has adapted one of his investigations to it, when treating of the pendulum.

Another consequence of this proposition is, that if the centre of the ellipse be supposed to be removed to an infinite distance, and the figure to become a parabola, the centripetal force being directed to a point infinitely remote, becomes constant and equable; a proposition discovered first by Galileo.

Sir Isaac Newton having treated of the centripetal force in conic sections, where the centre of forces is the centre of the figure, (and generally whatever be the centre in the case of the circle,) proceeds to treat of that force where it is directed towards the focus of one or other of those curves, and not to the centre. It is easy to demonstrate a compendious theorem, that which forms the subject of his three first propositions, in which he determines the law of the force for the three curves (parabola, hyperbola, and ellipse) severally. For this purpose a simple reference to the formulæ already stated will suffice; indeed our illustration of those formulæ has already anticipated this.

If O P A be a conic section whose parameter is D, S Y the perpendicular to the tangent T P, P R the radius of curvature at P; then S Y : S P :: $\frac{1}{2}$ D : P N (the normal), and

$S\,Y = \frac{D.S\,P}{2.P\,N}$; also $P\,R = \frac{4\,P\,N^3}{D^2}$; substitute these values of S Y and P R (p and R) in the



expression formerly given for the central force $\frac{r}{2\,p^3 \times R}$, and we have $\frac{S\,P}{\frac{2\,D^3.\,S\,P^3}{8\,P\,N^3} \times \frac{4\,P\,N^3}{D^2}}$ or $\frac{1}{D \times S\,P^2}$, which is (D being invariable) the inverse square of the distance. Therefore any body moving in any of the conic sections by a force directed to the focus, is attracted by a centripetal force inversely as the square of the distance from that focus. This demonstration, therefore, is quite general in its application to all the conic sections.

It follows that if a body is impelled in a straight line with any velocity whatever, from an instantaneous force, and is at the same time constantly acted upon by a centripetal force which is inversely as the square of the distance from the centre, the path which the body describes will be one or other of the conic sections. For if we take the expression $\frac{1}{D.\ S\ P^2}$ and work backwards, multiplying the numerator and denominator both by S P, and then multiplying the denominator by $\frac{8\ D^2.\ P\ N^3}{8\ D^2.\ P\ N^3}$, we obtain the expressions for the value of S Y, the perpendicular, and for R, the radius of curvature. But no curves can have the same value of S Y and R, except the conic sections; because there are no other curves of the second order, and those values give quadratic equations between the co-ordinates. By pursuing another course of the same kind algebraically, we obtain an equation to the conic sections generally, according as certain constants in it bear one or other proportion to one another. The perpendicular S Y and the radius of curvature are given in terms of the normal; and either one or the other will give the equation. Thus

$$R = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx^2 \times d\left(\frac{dy}{dx}\right)} = \frac{4\,P\,N^3}{D^2} = \frac{4\,y^3}{D^2\,dx^3} \times (d\,x^2 + d\,y^2)^{\frac{3}{2}}$$

which gives $D^2\,d\,x^3 = 4\,y^3 \times (d^2\,y\,d\,x - d^2\,x\,d\,y)$

an equation to the co-ordinates. Now whether this be resolvable or not, it proves that only one description of curves, of one order, can be such as to have the property in question. The former operation of going back from the expression of the central force, proves that the conic sections answer this condition. Therefore no other curves can be the trajectories of bodies moving by a centripetal force inversely as the square of the distance.*

It may be remarked that J. Bernouilli objects (Mém. Acad. des Sciences, 1710) to Sir Isaac Newton that he had assumed the truth of this important proposition without any demonstration. But this is not correct. He certainly gives a very concise and compendious one; but he states distinctly that the focus and point of contact being given, and the tangent given in position, a conic section may be described which shall at that point of contact have a given curvature; that the curvature is given from the velocity and central force being given; and that two orbits touching each other with the same centripetal force and velocity

* The equation may be resolved and integrated; there results, in the first instance, the equation $d x = \frac{2 y d y}{\sqrt{2 c y^2 - D^2}}$, and therefore the fluent is this quadratic, $c^2 x^2 = 2 c y^2 - D^2 + C$.

cannot be described. This is in substance what we have expounded in the above demonstration. But it must also be observed, as Laplace has remarked, that Newton has in a subsequent problem shown how to find the curve in which a body must move with a given velocity, initial direction, and position; and since, when the centripetal force is inversely as the square of the distance, the curve is shown to be one or other of the conic sections, he has thus demonstrated the proposition in question; so that if he had not done so in the corollary to one problem, he has in the solution of another.*

J. Bernouilli objects also to a very concise and elegant solution of the inverse problem given by Herrman in the same volume of the Mémoires, and which had been communicated to him before it was presented to the Academy. This solution proceeds upon his general expression for the centripetal force of $-\frac{d^2 x}{x} \sqrt{x^2 + y^2}$; and the objection made is that he works the problem (as he does in a few lines) by multiplications and divisions which show that he

* Système du Monde, liv. v. chap. 5. It is to be observed, that the Seventeenth Prop. Book I., is exactly the same in the first as in the subsequent editions, except the immaterial addition of a few lines to the demonstration. Consequently, Bernouilli must have been aware of it when he wrote in 1710.

was previously aware of the solution in the case of the conic sections. But this is no objection to a solution which being of a problem already known, can only be regarded as a demonstration that the former solution was exact. It is an objection which, if valid, applies certainly to the demonstration which we have just given of the proposition; but so it does to all the demonstrations of the ancient geometrical analysis. It is a more substantial objection that Herrman omitted a constant in his integration; but by adding it, Bernouilli shows that the equation which Herrman found, when thus corrected, expresses the conic sections generally.

This truth, therefore, of the necessary connexion between motion in a conic section and a centripetal force inversely as the square of the distance from the focus, is fully established by rigorous demonstration of various kinds.

If we now compare the motion of different bodies in concentric orbits of the same conic sections, we shall find that their motion, the areas which their radii vectores describe round the same focus, are to one another in the subduplicate ratio of the parameters of those curves. From this it follows, that in the ellipse whose conjugate axis is a mean proportional between its transverse axis and parameter, the whole time taken to revolve (or the periodic time) being in the proportion of the area (that is in the proportion of the rectangle of the axes) inversely, and in the subduplicate ratio of the parameter directly, is in the sesquiplicate ratio of the transverse axis, and equal to the periodic time in a circle whose diameter is that axis. It is also easy to show from the formula already given respecting the perpendicular to the tangent, that the velocities of bodies moving in similar conic sections round the same focus, are in the compound ratio of the perpendiculars inversely and the square roots of the parameters* directly. Hence in the parabola a very simple expression obtains for the velocity. For the square of the perpendicular being as the distance from the focus by the nature of the curve, (the former being $a^2 + a\,x$, and the latter $a + x$), the velocity is inversely as the square root of that distance. In the ellipse and hyperbola where the square of the perpendicular varies differently in proportion to the distance, the law of the velocity varies differently also. The square of the perpendicular in the ellipse (A being the transverse axis and B the conjugate, and r the radius vector) is

* By parameter is always to be understood, unless otherwise mentioned, the principal parameter, or the parameter to the principal diameter.

$\frac{B^2 \times r}{A - r}$; in the hyperbola, $\frac{B^2 \times r}{A + r}$, or those squares of the perpendicular vary as $\frac{r}{A - r}$ and $\frac{r}{A + r}$, in those curves respectively, B^2 being constant. Hence the velocities of bodies moving in the former curve vary in a greater ratio than that of the inverse subduplicate of the distance, or $\frac{1}{\sqrt{r}}$, and in a smaller ratio in the latter curve, while in the parabola $\frac{1}{\sqrt{r}}$ is their exact measure.

To these useful propositions, Demoivre added a theorem of great beauty and simplicity respecting motion in the ellipse. The velocity in any point P is to the velocity in T, the point where the conjugate axis cuts the curve, as the square root of the line joining the former point P and the more distant focus, is to the square root of the line joining P and the nearer focus. It follows from these propositions that in the ellipse, the conjugate axis being a mean proportional between the transverse and the parameter, and the periodic time being as the area, that is as the rectangle of the axes directly, and the square root of the parameter inversely, t being that time, a and b the axes, and

p the parameter, $t = \frac{a\,b}{\sqrt{p}}$, and $b^2 = a\,p$; therefore $a\,b = a\,\sqrt{a\,p} = \sqrt{a^3} \times \sqrt{p}$; and $t = \sqrt{a^3}$, and $t^2 = a^3$; or the squares of the periodic times are as the cubes of the mean distances. So that all Kepler's three laws have now been demonstrated, *à priori,* as mathematical truths; the areas proportional to the times, if the force is centripetal, and the elliptical orbit and sesquiplicate ratio of the times and distances, if the force is inversely as the squares of the distances, or in other words if the force is gravity.

Again, if we have the velocity in a given point, the law of the centripetal force, the absolute quantity of that force in the point, and the direction of the projectile or centrifugal force, we can find the orbit. The velocity in the conic section being to that in a circle at the given distance D, as m to n, and the perpendicular to the tangent being p, the lesser axis will be $\frac{2\,m\,p}{\sqrt{2\,n^2 - m^2}}$, and the greater axis $\frac{2\,\mathrm{D}\,n^2}{2\,n^2 - m^2}$, the signs being reversed in the denominator of each quantity for the case of the hyperbola. Hence the very important conclusion that the length of the greater axis does not depend at all upon the direction of the tangential or projectile force, but only upon its quantity, the direction influencing the length of the lesser axis alone.

Lastly, it may be observed, that as these latter propositions give a measure of the velocity in terms of the radius vector and perpendicular to the tangent for each of the conic sections, we are enabled by knowing that velocity in any given case where the centripetal force is inversely as the square of the distance, and the absolute amount of that force is given, as well as the direction of the projectile force and the point of the projection, to determine the parameters and foci of the curve, and also which of the conic sections is the one described with that force. For it will be a parabola, an hyperbola, or an ellipse, according as the expression obtained for p^2 (the square of the perpendicular to the tangent) is as the radius vector, or in a greater proportion, or in a less proportion. This is the problem above referred to, which John Bernouilli had entirely overlooked, when he charged Sir Isaac Newton with having left unproved the important theorem respecting motion in a conic section, which is clearly involved in its solution.

Before leaving this proposition, it is right to observe that the two last of its corollaries give one of those sagacious anticipations of future discovery which it is in vain to look for anywhere but in the writings of this great man.* He says, that by pursuing the methods indicated in the investigation, we may determine the variations impressed upon curvilinear motion by the action of disturbing, or, what he terms, foreign forces; for the changes introduced by these in some places, he says, may be found, and those in the intermediate places supplied, by the analogy of the series. This was reserved for Lagrange and Laplace, whose immortal labours have reduced the theory of disturbed motion to almost as great certainty as that of untroubled motion round a point by virtue of forces directed thither.†

We have thus seen how important in determining all the questions, both direct and inverse, relating

* See a singular anticipation respecting dynamics, by Lord Bacon, in *De Aug. Lib.* III., under the head Translation of Experiments. It was pointed out to me by my learned friend B. Montague.

† Laplace (Méc. Cél. lib. xv. ch. i.) refers to this remarkable passage as the germ of Lagrange's investigations in the Berlin Mémoires for 1786.

to the centripetal force, are the perpendicular to the tangent and the radius of curvature. Indeed it must evidently be so, when we consider, *first*, that the curvature of any orbit depends upon the action of the central force, and that the circle coinciding with the curve at each point, beside being of well-known properties, is the curve in which at all its points the central force must be the same; and, *secondly*, that the perpendicular to the tangent forms one side of a triangle similar to the triangle of which the fluxion of the radius vector is a side; the other side of the former triangle being the radius vector, the proportion of which to the force itself is the material point in all such inquiries. The difficulty of solving all these problems arises from the difficulty of obtaining simple expressions for those two lines, the perpendicular p and the radius of curvature R. The radius vector r being always $\sqrt{x^2 + y^2}$ interposes little embarrassment; but the other two lines can seldom be concisely and simply expressed. In some cases the value of F, the force, by $d\,r$ and $d\,p$ may be more convenient than in others; because p may involve the investigation in less difficulty than R; besides that p^3 enters into the expression which has no fluxions. But in the greater number of instances, especially where the curve is given, the formula $\frac{r}{2p^3 R}$ will be found most easily dealt with.

ii. The next branch of the inquiry relates to the describing the conic sections severally, where certain points are given through which they are to pass, or certain lines which they are to touch. The subject is handled in two sections, (the fourth and fifth,) the first of which treats the case where one of the foci is given; the second the case where neither focus is given. This whole subject is purely geometrical; and exhibits a fertility of resources in treating these difficult problems, as well as an elegance in the manner of their solution, which has few parallels in the history of ancient or modern geometry. This portion of the Principia, however, is incapable of abridgment; and there is no advantage whatever in resolving the problems analytically, but rather the contrary; for with the exception of one of the lemmas, in demonstrating which Sir Isaac Newton himself has recourse to algebraical reasoning in order to shorten the proofs, the geometrical process is in almost every instance extremely concise, in all cases much more beautiful, and less encumbered than the algebraical. The superiority of the former to the latter method of investigation in such solutions is apparent on trying algebraically some simple case as in the solution, describing a circle through three points, or through two points, and touching a line given in position; no little embarrassment results from the number and entanglement of the quantities in the solution. Even so great a master of analysis as Sir Isaac Newton, in solving the problem of describing a circle through two points, touching a given line, could find no better expression than

$x = \dfrac{-e^2 b \sqrt{e^2b^2 + e^2a^2 - d^2a}}{d^2 - a^2}$, although geometrically the construction is easy by drawing a circle on one segment of the line joining the given points, and another on the given line.* These are comparatively simple problems; in the more difficult cases of the conic sections this embarrassment is often inextricable.†

To illustrate the application of these important problems, let us suppose that by observation we

* The above algebraical solution is that of Prop. 43 of the Arith. Univ., where the 59,60 and 61 are also solutions of the three first problems of Sect. V. of the Principia, B. I.

† Maria Agnesi's Instituzioni Anatiliche abounds in elegant algebraical investigations of geometrical problems, but affords no grounds for modifying the above remark.

obtain three points in the orbit of any planet, and would ascertain from those points the position of the greater axis, and the focus in which the sun is placed, the eccentricity of the orbit or distance of the focus from the centre of the ellipse, and the aphelion, or greatest distance to which in its course the planet ever is removed from the sun; this is easily done by means of Prop. XVIII. (Book I.), for that enables us to find the elliptical and hyperbolical trajectories, which pass through given points, when one focus and the transverse axis are given; and thus to find the other focus, and the centre of the curve, and the distance from the given focus to the further extremity of the axis, which is the aphelion.

In like manner the problem which Sir Isaac Newton calls by far the most difficult of any, and says that he had tried to solve in various ways,* that of finding the trajectory of a comet from three observations, supposing it to move in a parabolic orbit, is reduced by an elaborate and difficult process of reasoning to describing a parabola through two given points, which are found in its own orbit from the observations. Now Prop. XIX. of Book I.

* Problema hocce longe difficillimum multimode aggressus (Lib. III. Prop. 41).

gives an easy solution of this problem.* It is only to describe from each of the given points a circle, with the distance of that point from the given focus as a radius, and the straight line touching these two circles will be the directrix of the parabola, and the perpendicular to it from the focus, its axis; the principal vertex being the middle point of that perpendicular. The coincidence of the very eccentric elliptical orbits of the comets with the parabola makes this parabolic hypothesis answer for determining their places and times in the general case.

The correction of the orbit thus found is reduced to finding the orbit of an ellipse which shall pass through three given points, and this is done by the 21st proposition of Book I., or rather by the 16th lemma, to which it is a corollary, for inflecting three straight lines from three given points, the differences, if any, between the lines, being given.

Sir Isaac Newton tried the accuracy of the methods thus found upon several comets, and parti-

* Several other propositions are given in the first book for the purpose of facilitating the solution of this difficult problem by another method; but the author informs us that he subsequently fell upon the method which he has given in the third book, and which he prefers for its greater simplicity.

cularly on the celebrated one of 1680, called Halley's comet, from the great labour which that mathematician, in aid of his illustrious friend and master, bestowed upon the calculation of its orbit. The following is a short statement of the general result of a comparison between the places computed from the theory, and the places found by actual observation, in the cases tried.

First, as regards the comet of 1680, or Halley's comet.

In comparing four observations with the geometrical computation, Sir Isaac Newton found an error of 5′ 3″ on an average in the latitude, and about 1′ in the longitude. But Halley, having afterwards made the computations with greater accuracy by arithmetical operations, found the average error, on sixteen observations, in the latitude only about 52″, and in the longitude 1′ 28″. The average error found on a comparison of the theory with twenty-one observations made abroad, was found by Halley only to be 50″ in the latitude, and 57″ in the longitude.*

Secondly, as regards other comets.

In the computations of the comet 1665, the error

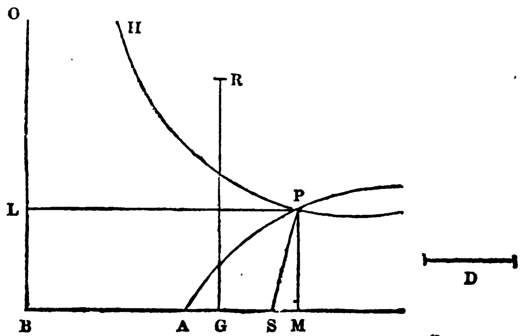
* This omits the observation made 26th December, as there is manifestly an error in the figures in that observation.

was, on an average of eighteen observations, 8″ in the latitude, and in the longitude 1′ 25″. In the latitude the errors by excess nearly balance those by defect, the one being to the other as 40 to 49. In the longitude, supposing the observation of December 7 accurately stated (which, from the error, amounting to 7′ 33″, seems very doubtful), the errors by excess are sixteen times more considerable than those by defect. In the comets of 1682 and 1683, on comparing the observations of Flamstead with the theory, the error was 1′ 31″ in latitude, and 45″ in longitude, for eleven observations of the former comet, and for seventeen of the latter comet, 1′ 10″ in latitude, and 1′ 29″ in longitude. But the comet of 1723 came nearer its computed place; the average error of latitude on fifteen observations of Bradley, compared with the same number by Halley himself, and Pound (his uncle), was only 21″½ in the latitude, and somewhat under 25″ in the longitude. It is to be remarked that is apparently the case in which the observations were the most accurate, three eminent observers checking each other, and no one observation differing from the computation much more than by the average of the rest, while great differences occur in all the other cases, and give rise to a suspicion of error. For in the comet of 1683, there was one day (Aug. 15) in which the latitude differed between three and four times, and the longitude three times more than the average; and in the observations of the comet of 1665 there are several errors in longitude of twice, and one error of no less than five times, above the average. These particular observations, and not the theory, then, were probably at fault in those instances; but they affect the general average materially.

The intimate connexion between the purely geometrical parts of the Principia, the Fifth and Sixth Sections of the First Book, and the most sublime inquiries into the motions of the heavenly bodies, those motions, too, which are the most rapid, and performed in spaces the most prodigious, may suffice to show the student how well worthy these mathematical investigations are of being minutely followed. Were they wholly unconnected with such important speculations in Physical Astronomy, and only to be regarded as a branch of the Higher Geometry, they would deserve the deepest attention, for their interesting development of general relations between figures so well known as the conic sections, for the marvellous felicity of the expedients by which the solutions are obtained, and for the inimitable elegance with which the reasoning is conducted. As a mere matter of mathematical contemplation, beginning and ending in the discovery of the relations which subsist between different quantities and figures, they afford matter of lasting interest to the geometrician. But it certainly heightens that interest to reflect that the same skilful and simple construction which enables us to describe a parabola through given points, or touching given lines, beside gratifying a curiosity purely geometrical, leads us to calculate within 20″ of the truth the place of bodies revolving round the sun in orbits so eccentric that the ellipse which they describe coincides with a parabolic line, instead of being nearly circular like the path of our globe, although our own distance from that luminary is near a hundred millions of miles.

iii. We are next to consider the motion of bodies in conic sections which are given, and ascending or descending in straight lines under the influence of gravity; that is, the velocities and the times of their reaching given points, or their places at given times. This branch of the subject, therefore, divides itself into two parts, the one relating to motion in the conic sections, the other to the motion of bodies ascending or descending under the influence of gravitation. The Sixth Section treats of the former, the Seventh of the latter.

(1.) In order to find the place of a revolving body in its trajectory at any given time, we have to find a point such that the area cut off by the radius vector to that point shall be of a given amount; for that area is proportional to the time. Thus suppose the body moves in a parabola, and that its radius vector completes in any time a certain space, say in half a year moves through a space making an area equal to the square of D; in order to ascertain its position in any given day of that half year, we have to cut off by a line drawn from the centre of forces an area which shall bear to D^2 the same proportion that the given time bears to the half year, say 3 to m^2, or we have to cut off a section A S P = $\frac{3}{m^2}$ D^2. A P being the parabola and S the focus;



this will be done if A B be taken equal to three times A S, and B O being drawn perpendicular to A B, between B O, BA asymptotes, a rectangular hyperbola is drawn, H P, whose semi-axis or semi-parameter is to D in the proportion of 6 to m; it will cut the parabolic trajectory in the point P, required. For calling A M $= x$ and P M $= y$ and A S $= a$; then A B $= 3\,a$ and $y \times (x + 3\,a)$ $=$ half the square of the hyperbola's semi-axis, which axis being equal to $\frac{6\,D}{m}$, $y\,(x + 3\,a) = \frac{36\,D^2}{2\,m^2}$ $= \frac{18\,D^2}{m^2}$, or $y\left(\frac{x}{3} + a\right) = \frac{6\,D^2}{m^2}$, and $y\left(\frac{x}{6} + \frac{a}{2}\right)$ $= \frac{3\,D^2}{m^2}$, or $y \times \left(\frac{2}{3}\,x - \frac{1}{2}\,x + \frac{1}{2}\,a\right) = \frac{3\,D^2}{m^2}$. Therefore $\frac{2}{3}\,x\,y - \frac{1}{2}\,(x - a)\,y = \frac{3\,D^2}{m^2}$, and $\frac{2}{3}$. A M $\times$ P M $= \frac{2}{3}\,xy$; and $\frac{1}{2}\,(x - a)\,y = \frac{1}{2}$ S M . P M $=$ S M P; therefore the sector A S P $= \frac{3\,D^2}{m^2}$: so that the radius from the focus S cuts off the given area, and therefore P is the point where the comet or other body will be found in $\frac{3}{m^2}$ parts of the time.

If the point is to be found by computation, we can easily find the value of y by a cubic equation, $y^3 + 3\,a^2\,y = \frac{18\,a^2\,D^2}{m^2}$, and making B L $= y$, L P parallel to A M, cuts A P in the point P required. Sir Isaac Newton gives a very elegant solution geometrically by bisecting A S in G, and taking the perpendicular G R to the given area as 3 to 4 A S, or to S B, and then describing a circle with the radius R S; it cuts the parabola in P, the point required.* This solution is infinitely preferable to ours by the hyperbola, except that the demonstration is not so easy, and the algebraical demonstration far from simple.

It is further to be observed, that the place being given, either of these solutions enables us to find the time. Thus in the cubic equation, we have only to find $\frac{3\,D^2}{m^2}$. It is equal to $\frac{y^3 + 3\,a^2\,y}{6\,a^2}$; and as D^2 is the given integer, or period of *e. g.* half a year, the body comes to the point P in a time which bears to D^2 the proportion of unity to $\frac{6\,a^2\,D^2}{y^3 + 3\,a^2 . y}$.

Sir Isaac Newton proceeds to the solution of the

* The most singular relation subsists between the hyperbolas and parabolic areas, giving rise to very curious Porisms connected with Quadratures.—See *Phil. Trans.* 1798, part ii.

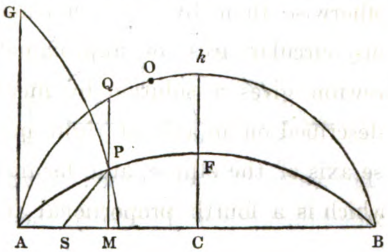
same important problem in the case of the ellipse, which is that of the planetary system, and is termed Kepler's problem from having been proposed by him when he had discovered by observation that the planetary motions were performed in this curve, and that the areas described by the radii were proportional to the times. In the parabola which is quadrable and easily so, the area being two-thirds of the rectangle under the co-ordinates, the solution of this problem is extremely easy. But the ellipse not admitting of an expression for its area, or the area of its sectors, in finite terms of any product of straight lines, the problem becomes incapable of a definite solution. Newton accordingly begins his investigation by a lemma, in which he endeavours to demonstrate that no figure of an oval form, no curve returning into itself and without touching any infinite arch, is capable of definite quadrature. It is rarely, indeed, that the expression "endeavour," can be applied to Sir Isaac Newton. But some have questioned the conclusiveness of his reasoning in this instance. The demonstration consists in supposing a straight line to revolve round a point within the oval, while another point moves along it with a velocity as the square of the portion of the revolving line between the given centre and the oval, that is, as the radius vector of the oval from the given centre. It is certainly shown, that the moving point describes a spiral of infinite revolutions; and, also, that its radius is always as the area of the oval at the point where that radius meets the oval. If then the relation between the area and any two ordinates from the oval to any axis is such as can be expressed by a finite equation, so can the relation between the radius of the spiral and co-ordinates drawn parallel to the former, or the co-ordinates to the same axis. Therefore it will follow, that the spiral can be cut only in a finite number of points by a straight line, contrary to the nature of that curve. Indeed, its co-ordinates being related to each other by an algebraical equation is equally contrary to its nature; consequently the possibility of expressing the relation between the area of the oval and the co-ordinates leads to this absurd conclusion, and therefore that possibility cannot exist; and hence it is inferred that the oval is not quadrable.

Sir Isaac Newton himself observes that this demonstration does not apply to ovals which form parts of curves, being touched by branches of infinite extent. But it does not even apply to all cases of ovals returning into themselves, and unconnected with any infinite branches. There is, for example, a large class of curves of many orders, those whose equation is $y^m = n^m x^{(n-1)m} \times (a^n - x^n)$; and when m is even these curves are quadrable; and in every case where m and n are whole positive even numbers, it is the equation to a curve returning into itself. This is manifest upon inspection: for $\int y\,d\,x = \int n\,x^{n-1}\,(a^n - x^n)^{\frac{1}{m}}\,d\,x$ is integrable because the power of x without is one less than that of x within the radical sign; and because there is no divisor there can be no asymptote; while it is plain that the $\frac{1}{m}$ root of $a^n - x^n$ is impossible when either $+x$ or $-x$ is greater than a, n and m being both whole numbers and m even. Wherefore the curve returns into itself; and as $y = 0$, both when $x = 0$, and when $x = +a$, or $-a$, therefore the figure consists of two ovals meeting or touching in the origin of the abscissæ. These two ovals admit of a perfect quadrature; the integral being $C - \frac{m}{m+1}\,(a^n - x^n)^{\frac{m+1}{m}}$. Thus if $m = n = 2$ the area is $C - \frac{2}{3}\,(a^2 - x^2)^{\frac{3}{2}}$, the latter quantity being an area that has to one-third the rectangle of the co-ordinates the same proportion which the difference of the squares of the diameter and abscissa has to the square of the abscissa; for $\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}} = \frac{1}{3}xy \times \frac{a^2 - x^2}{x^2}$.

The particular inquiry respecting motion in the ellipse did not perhaps require the proposition to be proved in the very general form in which Sir Isaac Newton has given it. That the ellipse cannot be squared might perhaps be sufficiently proved from this consideration, founded upon a reasoning analogous to that on which the lemma in question proceeds. If a curve be described such that its co-ordinates, or the rectangle contained by the co-ordinates, shall always bear a given proportion to the areas of the ellipse on the same axis, this curve cannot be algebraical, not merely because of its equation involving quantities not integrable (for that may be said to be the question), but because it will stop short at a given line, which no algebraical curve can do. It will have no branch extending beyond the perpendicular at the end of the axis: and moreover its equation is known to be that of a transcendental curve. This reason cannot be applied to all curves returning into themselves; because, as we have seen in one class, the equation to the curve, whose co-ordinates should express their areas, is algebraical; and also because, in that class, the secondary curve is found to have two branches which meet in cusps, and so do not stop short. If described by the proportion of areas they would seem to stop short, that property only belonging to one of their branches; but their equation discloses the second branch. It is one of many instances of a truth perhaps not sufficiently remarked by geometricians, that curves sometimes have particular portions to which certain properties belong exclusively, no other part of the curve having them.

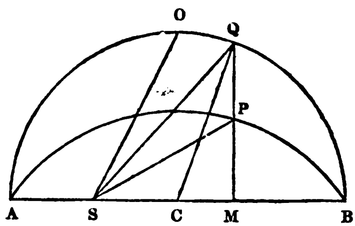
As the area of the ellipse cannot be found by algebraical quantities, or by the description of algebraical curves, the problem of Kepler cannot be solved otherwise than by transcendental curves, logarithms, circular arcs, or approximation. Sir Isaac Newton gives a solution by means of the cycloid described on an axis at right angles to the transverse axis of the ellipse, at a distance from its vertex which is a fourth proportional to half the transverse axis, the focal distance, and the eccentricity, and with a generating circle whose radius is the distance of this perpendicular from the centre.

A parallel to the cycloid's axis, at the point whose abscissa is to the periphery of the generating circle in the proportion of the given time to the periodic time, cuts the ellipse at the place required. This solution requires a construction beside that of the curve described; but a cycloid may be described which shall cut the ellipse directly at the point required. If a circle is described on A B the transverse axis, and its quadrant A k is cut in O, in the given ratio of the times in which the elliptical area is to be cut; and then a cycloid is described, whose ordinate P M is always a fourth proportional to the arch O Q, the rectangle of the two axes and the distance between the foci, or to A B × 2. C F, and 2. C S, —the cycloid cuts the ellipse in the point required, P. The equation to this curve G P is simple enough,



and the construction easy, for the ordinate is in a given proportion to the arc Q O of the quadrant. As, however, an arithmetical approximation by means of series is required in practice, Sir Isaac Newton gives two methods, both of great elegance and efficiency.

It may be proper here to note the names given by astronomers to the lines and angles in the ellipse connected mainly with the investigation of this problem. The sun being in the focus S, and P the planet's place, the aphelion of the planet is B; the perihelion A; the arch B P, or angle B S P is the true anomaly; B O being to the whole circumference as the time in B P to the whole periodic time, B O, or O S B, is the mean anomaly, and Q B, or Q C B, is the eccentric anomaly,

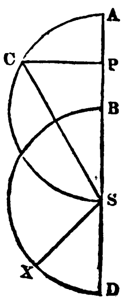


C being the centre of the ellipse: A and B are likewise called the apsides (or apses), and A B, the transverse axis, is called the line of the apsides; S C, or more generally $\frac{SC}{AC}$ is the eccentricity.

(2.) The next subject of inquiry is the comparison of bodies moving in a straight line towards the centre of forces, with those moving by the same centripetal force in the conic sections whose axis is that straight line. If the projectile force by which a body revolves in any of those curves round the focus as a centre, suddenly ceases, and the body falls towards the centre of the curve, it is shown that its place at any given time will be the point where the line of descent is cut by a perpendicular from the point of the curve where the radius from the vertex makes its area proportioned to the time consumed in the fall. For take the parabola whose area is $\frac{2}{3}\,x\,y$, and let the distance of the point where the body begins to descend in a straight line be c; the parabolic sectors, which are as the times, are expressed by $y \times \left(\frac{x + 3\,c}{6}\right) \left(= \frac{2}{3}\,x\,y + (c - x)\,\frac{y}{2}\right)$ or $\frac{\sqrt{a\,x}}{6} \times (x + 3\,c)$; and if another parabola with the same vertex, and with a smaller parameter, b, is drawn nearer the straight line, its sectors are $\frac{\sqrt{b\,x}}{6}\,(x + 3\,c)$. Now the times in the first parabola, or the areas, at any two points referred to the abscissæ x and z, being $\frac{\sqrt{a x}}{6}$ $(x + 3 c)$, and $\frac{\sqrt{a z}}{6}$ $(z + 3 c)$, the times or areas in the second parabola will be $\frac{\sqrt{b x}}{6}$ $(x + 3 c)$, and $\frac{\sqrt{b z}}{6}$ $(z + 3 c)$, respectively; and therefore it is evident that the areas at the distances x and z, in the one curve, are in the same proportion to one another with the areas in the other curve at those distances. If the parameter be continually diminished of the second curve, until that curve coincides with the axis, the same proportion holds; and the times, therefore, in falling through the axis, will be as the areas of the first curve, corresponding to the points of that axis: And so it may be shown in the ellipse and hyperbola.

Hence it follows, that in the case of the parabola, the velocity of the falling body in any given point is equal to that with which the body would, moving uniformly, describe a circle described from the centre, to which the body is falling, and with a diameter equal to the distance of the given point from that centre. In the circle, the velocity at the given point is to the velocity in the circle described from the centre, with the distance of the given point for the radius, as the square root of the distance fallen through to that of the whole distance of the point where the fall begins. Thus let d be the distance of the given point to which the body has fallen, D the distance of the point at which it began to fall; the velocity in the case of a parabola is equal to that of the body moving in a circle, whose radius is $\frac{1}{2}d$; in the case of a circle, it is to that of a body moving in a circle whose radius is d, as $\sqrt{D - d} : \sqrt{D}$; and the like proportion subsists in the case of the hyperbola.

Further, a rule is thus deduced for determining, conversely, the time of descent, the place being given. A circle is to be described on A S $=$ D, as the diameter, and another from S the centre, towards which the body falls, with the radius $\frac{D}{2}$. P being the point to which it has fallen, if the area S X B be



taken equal to S C A, the time taken to fall through A P is equal to the time in which the body would move uniformly from B to X. Hence the periodic times being in the sesquiplicate ratio of the distances ($t = d^{\frac{3}{2}}$) and because $2^{\frac{3}{2}} = 2\sqrt{2}$, the time taken to fall through the whole distance to the centre is to the periodic time of a body revolving at twice that distance round the same centre as 1 to $4\sqrt{2}$; and thus we can calculate the time (supposing the planetary orbits to be circular) which any one would take to fall in a straight line to the sun, or any satellite to its principal planet, if the projectile motion were suddenly to cease. The moon in this way would fall to the earth in about four hours less than five days.*

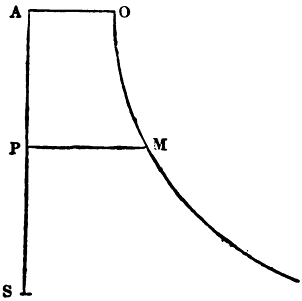
The inquiry is closed with a solution of the general problem, of which the preceding solutions for the conic sections, and for the force inversely as the squares of the distances, are only particular cases; and the times and velocities are found from the places,

* It is comparing the greatest with the smallest things, to observe that the time of the revolution of a planet round the sun, or the planetary year, bears the same proportion to the time in which the planet would fall to the sun, which the square of the side of a bees' cell does to one of the six triangles, or to the sixth part of the rhomboidal plate. (See Appendix to vol. i.)

or the places from the times and velocities, where a body ascends from or descends to the centre influenced by a centripetal force of whatever kind. On the given straight line of ascent or descent a curve is to be described whose co-ordinates are the centripetal force at each point of the axis, or whose equation is $y=X$, X being a function of x, the distance from the beginning of the motion. The area of the curve at each point is $\int y\,dx=\int X\,dx$; and if that fluent is equal to Z^2, Z is the velocity at the distance $a-x$, from the centre. Another curve described on the same axis, and whose equation is $u=\frac{1}{Z}$, gives by its areas $\int\frac{dx}{Z}=\zeta$, the time taken to move through the distance $a-x$; it is equal to ζ. This is easily demonstrated; for, first, if the velocity be v, and the time dt, the space being dx, we have the force $y=\frac{dv}{dt}$, and $dt=\frac{dx}{v}$, therefore $y=\frac{v\,dv}{dx}$, and $y\,dx=v\,dv$, and $\int y\,dx$ $=\frac{v^2}{2}$; but $Z^2=\int y\,dx$; therefore $Z=\frac{v}{\sqrt{2}}$, and the velocity is as the area Z. Again; for the time in the other curve; $u=\frac{1}{Z}$, and $v=\sqrt{2}.Z$; also $dt=\frac{dx}{v}=\frac{dx}{\sqrt{2}.Z}$. Therefore $t=\sqrt{2}\int\frac{dx}{Z}=$

$\sqrt{2}.\zeta$, or the time is as the area ζ. In these expressions, therefore, to find Z and ζ we have to substitute the values of X and Z in terms of x, and integrate.

It is hardly necessary to add, that if, instead of the velocity and the time being sought (Z and ζ), these are given, and the place reached by the body be sought, we find it by the same construction; and ascertaining what value of x gives the value of Z, the square root of the area. But it may be well to note here, that if O M be the curve, whose



ordinate P M or $y = X$, the centripetal force at P in terms of A P or x, or the gravitation of any particle of a homogeneous fluid towards S at the point P; then the column of that fluid whose altitude is A P will press at P, as the area A P M O, or as v^2, the square of the velocity acquired by a body falling through A P.

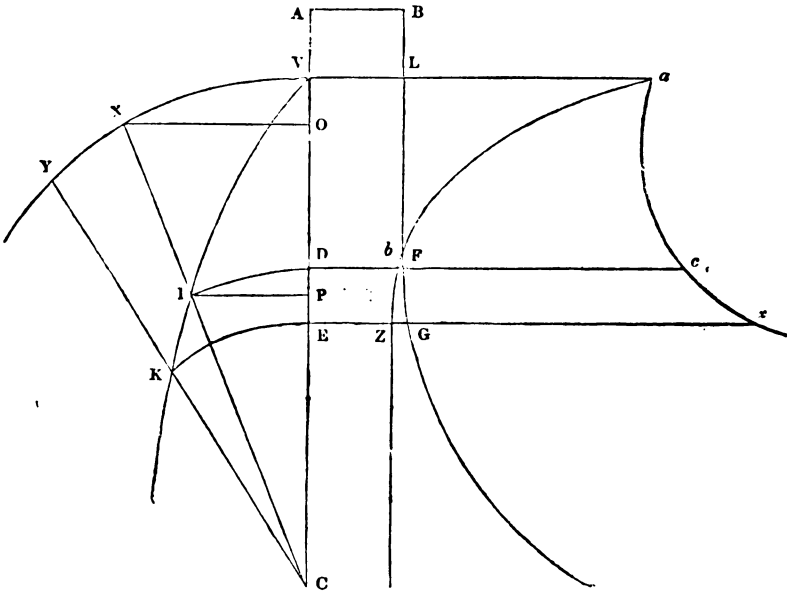
iv. The next object of research is to generalise the preceding investigations of trajectories from given forces, and of motion in given trajectories, applying the inquiry to all kinds of centripetal force, and all trajectories, instead of confining it to the conic sections, and to a force inversely as the square of the distance. This forms the subject of the Eighth Section, which therefore bears to the Third, Fourth, Fifth, and Sixth, the same relation that the concluding investigation of the Seventh Section (on rectilinear motion influenced by centripetal force) bears to the rest of that section.

The length at which we before went into the solution of the problem of central forces (inverting somewhat the order pursued in the Principia) makes it less necessary to enter fully into the general solution in this place. We formerly gave the manner of finding the force from the trajectory in general terms, and showed how, by means of various differential expressions, this process was facilitated. It must, however, be remarked, that the inverse problem of finding the trajectory from the force is not so satisfactorily solved by means of those expressions. For example, the most general one at which we arrived of $\frac{\sqrt{y^2+(x-a)^2} \times dx^2 . dX}{2\,(y\,dx-(x-a)\,dy)^3}$ being put

$=\frac{C}{y^2+(x-a)^2}$, or the force inversely as the square of the distance, presents an equation in which it may be pronounced impossible to separate the variables so as to integrate, at least while dX, the fluxion of $\frac{dy}{dx}$, remains in so unmanageable a form; for then the whole equation is $\frac{d^2y\,dx-d^2x\,dy}{2\,(y\,dx-(x-a)\,dy)^3}$

$= \frac{C}{(y^2+(x-a)^2)\,^{\frac{3}{2}}}$, and thus from hence no equation to the curve could be found. It cannot be doubted that Sir Isaac Newton, the discoverer of the calculus, had applied all its resources to these solutions, and as the expressions for the central force, whether $\frac{r}{2p^3.R}$, or $\frac{dp}{p^3dr}$, or $-\frac{d^2x\sqrt{x^2+y^2}}{x}$ (in some respects the simplest of all, being taken in respect of dt constant, and which is integrable in the case of the inverse squares of the distances, and gives the general equation to the conic sections with singular elegance), are all derivable from the Sixth Proposition of the First Book, it is eminently probable that he had first tried for a general solution by those means, and only had recourse to the one which he has given in the Forty-first Proposition when he found those methods unmanageable. This would naturally confirm him in his plan of preferring geometrical methods; though it is to be observed that this investigation, as well as the inverse problem for the case of rectilinear motion in the preceding section, is conducted more analytically than the greater part of the Principia, the reasoning of the demonstration conducting to the solution and not following it synthetically.

A is the height from which a body must fall to acquire the velocity at any point D, which the given body moving in the trajectory V I K (sought by the investigation) has at the corresponding point I; D I, E K, being circular arcs from the centre C, and C I$=$C D and C K$=$C E. It is shown previously that, if two bodies whose masses are as their weights descend with equal velocity from A, and being acted on by the same centripetal force, one moves in V I K and the other in A V C, they will at any corresponding points have the same velocity, that is at equal distances from the centre C. So that, if at any point D, D *b* or D F be as the velocity at D of the body moving in A V C, D *b* or D F will also represent the velocity at I of the body moving in



V I K. Then take D F $= y$ as the centripetal force in D or I (that is, as any power of the distance D C, or $a - x$, V C being a, and V D, x) V D F L will be $\int y \, d x$. Describe the circle V X Y with C V as radius. Let V X $= z$, and Y X will be $d z$, and N K $= \frac{x \, d z}{a}$. Then I C K being as the time, and $d t$ being constant, that triangle, or $\frac{\text{I C} \times \text{K N}}{2}$, is constant, and K N is as a constant quantity divided by I C, or as $\frac{Q}{x}$. If we take $\frac{Q}{x}$ to $\sqrt{\text{A V L B}}$ (proportioned to the force at any one point V and therefore given), as K N to I K, therefore this will in all points be the proportion; and the squares will be proportional, or $\int y\,d\,x : \frac{Q^2}{x^2} :: I K^2$, or $K N^2 + I N^2$, to $K N^2$; and therefore $\int y\,d\,x - \frac{Q^2}{x^2} : \frac{Q^2}{x^2} :: I N^2$, or $d\,x^2 : \frac{x^2\,d\,z^2}{a^2}$. Therefore $\frac{x\,d\,z}{a}$

$$= \frac{Q\,d\,x}{x\sqrt{\int y\,d\,x - \frac{Q^2}{x^2}}};$$ and multiplying by x, $\frac{x^2\,d\,z}{a}$

(twice the sector I C K) $= \frac{Q\,d\,x}{\sqrt{\int y\,d\,x - \frac{Q^2}{x^2}}}$. Again

$a\,d\,z : \frac{x^2\,d\,z}{a} :: a^2 : x^2$; and $a\,d\,z = \frac{x^2\,d\,z}{a} \times$

$\frac{a^2}{x^2} = \frac{a^2}{x^2} \times \frac{Q\,d\,x}{\sqrt{\int y\,d\,x - \frac{Q^2}{x^2}}}$ = twice the sector

Y C X. Hence results this construction. Describe the curve $a\,b$ Z, such that $(D\,b = u)$ its equation

shall be $u = \frac{Q}{2\sqrt{\int y\,d\,x - \frac{Q^2}{x^2}}}$, and the curve

$a\,c$ x such that $(D\,C = \varphi)$ its equation may be

$\varphi = \frac{Q\,a^2}{2\,x^2\sqrt{\int y\,d\,x - \frac{Q^2}{x^2}}}$. Then the fluxions of the areas of these curves, or $u\,d\,x$ and $\varphi\,d\,x$, being respectively $\frac{Q\,d\,x}{2\sqrt{\int y\,d\,x - \frac{Q^2}{x^2}}}$ and $\frac{Q\,a^2\,d\,x}{2\,x^2\sqrt{\int y\,d\,x - \frac{Q^2}{x^2}}}$, and these being equal to $\frac{x^2\,d\,z}{2\,a}$ and $\frac{a\,d\,z}{2}$, or the sectors which are the fluxions of the areas V I C and V X C, the areas themselves are equal to those areas; and therefore from V X C being given (if the area c D V a be found), and the radius C V being given in position and magnitude, the angle V C X is given; and from C X being given in position, and C V in magnitude and position, and the area C I V, if V D b a be found, the point I is found, and the curve V I K is known. This, however, depends upon the quantities made equal to u and φ severally being expressed in terms of x, for this is necessary in order to eliminate y from the equations to these curves; and then it is necessary to integrate these expressions; for else the angle V C X, and the curve V I K, are only obtained in fluxional equations. Hence Sir Isaac Newton makes the quadrature of curves, that is, first the integration of $\int y\,d\,x$, to eliminate y, and then the integration of the equations resulting in terms of u and

x, ϕ and x respectively, the assumptions or conditions of his enunciation. The inconvenience of this method of solving the problem gave rise to the investigations of Hermann and Bernouilli. The equation of the former, involving, however, the second fluxion of the co-ordinate, is to the rectangular co-ordinates; that of the latter is a polar equation, in terms of the radius vector and angle at the centre of forces.

To illustrate the difficulty with which this method of quadratures is applied, in practice—take the case of the centripetal force being inversely as the cube of the distance; then $y = \frac{1}{x^3}$ and the curve B L F is quadrable. If we seek the circle V X Y by rectangular co-ordinates X O, O C, we find the equation to obtain O C = D in terms of x, is of the form

$$\frac{2\,(a - \mathrm{D})^2\, d\,\mathrm{D}}{\sqrt{2\,a\,\mathrm{D} - \mathrm{D}^2}} = \frac{2\,a^2\,d\,x}{2\,x^2\sqrt{\int y\,d\,x - \frac{\mathrm{Q}^2}{x^2}}}$$

$$= \frac{a^2\,d\,x}{x\sqrt{2\,c\,x^2 - b - 2\,\mathrm{Q}^2}}$$

(b being the constant introduced by integrating $\int y\,d\,x$). Now there is no possibility of integrating these two quantities otherwise than by sines, and we thus obtain, nor can we do more, the following equation to D in terms of x;

$$\frac{x}{2} - \frac{3\,a}{2}\sqrt{2\,a\,\mathrm{D} - \mathrm{D}^2} + \frac{3\,a^2}{4} \times 2\,\text{arc sin.}\sqrt{\frac{\mathrm{D}}{2\,a}}$$

$$= \frac{a^2}{\sqrt{2\,\mathrm{Q}^2 + b}} \times \text{arc cos.}\left(\sqrt{\frac{2\,\mathrm{Q}^2 + b}{2\,c}} \times x\right).$$

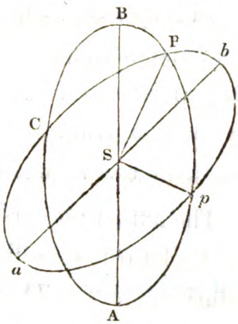
And suppose we could get D from this, in terms of cos. x, we have then to obtain P C by similar triangles, and then by another integration to obtain P I, in order to have the curve V I K.

But if we proceed otherwise, and instead of working by quadratures, take v the velocity of the body at I, or in the straight line at D, and make c the area described in a second, and θ the angle V C I, we obtain as a polar equation to V I K, $d\,\theta = \dfrac{c\,d\,x}{x\,\sqrt{4\,x^2\,v^2 - c^2}}$ (x being in this case both C D and the radius vector). Then, to apply this general equation to the case of the centripetal force being as $\dfrac{1}{x^3}$, let the force at the distance 1 be put equal to unity, and supposing the velocity of projection to be that acquired in falling from an infinite height, the equation to the trajectory becomes

$$d\,\theta = \frac{c\,d\,x}{x\,\sqrt{4 - c^2}}, \text{ and integrating, } \theta = \frac{c}{\sqrt{4 - c^2}} \times \log.\frac{x}{a}.$$

The whole subject of centripetal forces, inverse and direct, under the four heads which we began by stating, is therefore discussed, but always upon the assumption that the bodies acted upon move in orbits which remain at rest, and thus that the axis of the curve which they describe remains constantly in the same position. Another subject of inquiry is presented to us if that axis itself moves, revolving round the centre of forces, and we are required to ascertain the line in which the body moves in this moving orbit, as related to the line described by a body in a fixed orbit, or conversely to ascertain the motion in the two orbits. This subject divides itself into two branches, according as the planes in which the motions are performed pass through the centre of forces or not. Motions in the planes of the centre form the subject of the Ninth Section; the Tenth treats of motions in eccentric planes. Under the former division, a principal object of investigation is that which indeed measures the orbit's motion, and is identical with it, the motion of the apsides; that is, the position successively taken by the two points of the revolving orbit, where the tangents are perpendicular to the axis, and where, consequently, the moving body begins to come back towards the centre from its greatest distance in that direction of the axis; while, under the latter division of the subject, a main point of discussion is the vibration of pendulums.

i. If a body, revolving round a centre of forces, is acted upon laterally by any other force beside the centripetal and the centrifugal (or tangential), though the centre may remain fixed, the orbit will not remain so. The axis of the curve described will move forward or backward, according to the direction of the disturbing force. This motion of the axis is considered as a revolving motion of the orbit, and is the subject of our present consideration. The great practical importance of the inquiry will presently be shown. Suppose a body moves in an ellipse, the centripetal force being inversely as the square of the distance; the centrifugal force is in the direct proportion of the square of the velocity and the inverse proportion of the distance, jointly; that is, (a being the distance, and v the velocity in a circle,) as $\frac{v^2}{a}$; and v being as $\frac{1}{a}$, the centrifugal force is as $\frac{1}{a^3}$, or inversely as the cubes of the distances. A C B is the fixed ellipse; a C b, the one described by the body under the influence of a disturbing force, or in any other way made to move in an orbit whose axis, b S, or line joining the apsides A, B and a, b, is revolving round S. Sup-



pose the angular motion of the second ellipse to be in a given proportion to the motion of the body in the first, or that $s\ p$ being equal to S P, the angle B S P is $\frac{m}{n}$ of the angle b S p. The difference of the centrifugal forces of the two bodies must be equal to the difference of their centripetal forces. Calling T and t the centrifugal forces in the fixed and moveable orbits respectively; C and c the centripetal forces; $T - t = C - c$, and $c = C + t - T$. But $T : t$ in the proportion of the squares of the velocities, or of the angular motions, that is as $m^2 : n^2$ and $T - t : t :: m^2 - n^2 : n^2$, and because the centrifugal forces are at different distances, inversely as the cubes of those distances, therefore the difference of those forces in the two orbits, being in a given ratio to either of them, must be inversely as the cube of their common distance from the centre, or of the altitude of the revolving body in its orbit. Hence it follows that d being the common altitude or distance, and P the parameter, the force required to move the body in the moveable ellipse is as $\frac{m^2}{d^2} \times \frac{P(n^2 - m^2)}{2\,d^3}$; and, conversely, if such is the force, the motion will be in a moveable ellipse: and again, if a be the transverse axis of the ellipse, the forces in the fixed and in the moveable orbit will be to each other as $\frac{m^2\,d}{a^3}$ and $\frac{m^2\,d}{a^3} + P\,\frac{(n^2 - m^2)}{2\,d^3}$. Hence, in order that a body may move in a moveable ellipse, or an arc which advances or moves round in the direction of the body's motion, the centripetal force must vary in a higher proportion than the inverse square of the distance, but less than the cubes; and that the body may move in a retiring ellipse, or an arc which moves round in a direction contrary to that of the body, the centripetal force must vary in a less proportion than the inverse square of the distance.

From these propositions, Sir Isaac Newton is enabled to ascertain the proportion of the centripetal force to the distance, when the motion of the elliptical axis, that is of the apsides, or extreme points of it, shall be given, and conversely to ascertain the motion of the apsides when the proportion of the centripetal force to the distance is given. Let $\varphi : \theta$ be the proportion of the angular motion by which the body in the moveable orbit comes round to the same lines of apsides, to the angular motion of one revolution, or 360°; then the centripetal force will be as the power of the distance d, which is represented by $\frac{\varphi^2}{\theta^2} - 3$. Thus, if $\varphi = \theta$, or the axis of the moveable orbit moves only through the same space with the axis of the fixed orbit, that is if the moveable orbit coincides with the fixed, then the centripetal force is as $d^{1-3} = d^{-2} = \frac{1}{d^2}$, and conversely, if the central force is as $\frac{1}{d^2}$, the line of the apsides has no motion whatever. Hence the important proposition, that the inverse square of the distance, the actual law of gravitation, is the only proportion which prevents the line of the apsides from moving at all. Again, if $\varphi : \theta :: 363 : 360$, or the line of the apsides advances three degrees in each revolution, then the centripetal force is between the inverse square and inverse cube of the distance, but much nearer the former, for $\frac{\theta^2}{\varphi^2} - 3$ becomes nearly equal to $2\frac{4}{248}$, or about $2\frac{1}{61}$. But suppose the excess of the angle between the axes in one orbit over that angle in the other orbit to be only 11″ 53‴,* then $\frac{\theta^2}{\varphi^2} - 3$ becomes equal to $-2\frac{1}{59936}$, or $-2\frac{1}{60000}$, and the force as $\frac{1}{d^2 + \frac{1}{60000}}$. In like manner, if some extraneous force is impressed upon the revolving body, from knowing the amount of that force we can find the motion of the apsides, and conversely. It is found by following the method of Sir Isaac Newton, that the advance in a single revolution on the supposition of the disturbing force being to the centripetal force as 1 to 357·45, is equal to 1° 31′ 28″.

* The amount of 12″ is often given for the advance of the axis of the earth's orbit; but we have followed Laplace's number of 36″ 7‴, which on the sexagesimal scale is 11″ 89, or 11″ 53‴. This small difference makes a difference of 1000 years in the total revolution.

Now it is found that in the planetary motions these variations of the centripetal force actually take place. The action of the sun, for example, upon the moon, while she is acted upon by the earth in some parts of her orbit, coincides with that of the earth, and in some parts opposes this action, alternately adding to and taking away from the force of her gravitation towards the earth; and this increase and diminution is greater at the greater distances of the moon from the earth. Hence the proportion of the centripetal force which keeps her in her orbit is somewhat different from the exact ratio of the inverse square of the distance. There is more taken away from this centripetal force by the sun's action while the bodies are placed towards each other in one direction, than there is added when in the other position, and therefore there is a total diminution of the moon's gravitation, or the centripetal force decreases in a somewhat higher ratio than as the square of the distance increases; that is the denominator of the expression $\frac{1}{d^2}$ is greater than this exact power of d, which we have seen keeps the orbit and its axis fixed with respect to the centre, which in this case is the centre of the earth.

Hence this axis of the moon's orbit revolves in the direction of the moon's motion, and in a certain period makes a complete revolution; so that at one time, half this period, the moon's greatest and least distances from the earth (her apogee and perigee) have changed places, and at the end of the period they resume their former position. The amount of this motion of her apsides is about 3° in each revolution, or 39 in a year; so that the axis of her ellipse revolves in nine years; and the centripetal force is not as $\frac{1}{d^2}$ but $\frac{1}{d^{2\frac{1}{61}}}$, nearly the proportion above shown to belong to a progression of the apsides, equal to 3° in a revolution. In like manner the orbit of the earth is not immoveable, owing to the disturbing forces of the larger planets, Jupiter, Saturn, Mars, and Venus. But the disturbance here is, of course, incomparably more minute. The apsides of the earth's orbit only move 11″ 53‴ in the year, instead of 39°; and the expression for the centripetal force is therefore, as we have seen above, the inverse not of $d^{2\frac{1}{60}}$ but of $d^{2\frac{1}{60,000}}$. The axis of the earth's orbit thus resolves in a period of about 109,060 years.

It is, however, to be observed that, although this motion of the axis of the earth's orbit is the result of the theory of gravitation, and indeed affords a new proof of it, Sir Isaac Newton did not himself consider it as worthy of attention. He regarded it as indicating so very minute a deviation from the law of the inverse square of the distance, as not to alter sensibly the form and position of the orbits resulting from thence. He therefore did not give any calculation respecting it. To say that he was ignorant of it, or that he affirmed the absolute quiescence of the planetary apsides, as some have done,* is wholly erroneous. The statements and methods in the Forty-fifth proposition and its corollaries are quite general, applying to all bodies acted on by disturbing forces; so is the Sixty-seventh, with the Sixth, Seventh, and Eighth corollaries, of general application; and even in the proposition (the Fourteenth of the Third book) in which he affirms that the aphelia and nodes of the orbits are at rest, he refers to inequalities arising from disturbing forces, while in the scholium that immediately follows he expressly states the motion of the aphelion of Mars, and collects from thence that of the Earth, Venus, and Mercury, by the law which regulates the motion of the apsides,

* Bailly, Hist. Ast. tom. ii.

namely, the sesquiplicate proportion to the distances. By this he makes the motion of the Earth's aphelion 17′ 40″ in a century, or 10″ 36‴ yearly, being not a second and a half different from what it is now understood to be.

The calculation of the motion of the moon's apsides, however, which he deduced from these propositions, differed widely from the truth. He made it, as we have seen, amount to little more than a degree and a half each revolution,* or about one-half of the truth; and for the discrepancy between the theory and the phenomena he seems to have failed in accounting. Others, in the earlier part of the eighteenth century, having applied to the subject a different investigation, but founded upon his principles, obtained a different result, but erring by excess, for they made the motion 3° 27′ each revolution, or nearly 45° in the year instead of 39°. About the year 1745 the three great mathematicians of that age, Clairaut, Euler, and D'Alembert, investigated the subject; and, applying the whole resources of analysis to its discussion as a case of the problem of these bodies, obtained general solutions of great beauty. However they still found the theory differ with the fact nearly as much as New-

* 1° 31′ 28″:

ton himself had done; and Clairaut was even driven by this to devise a new law for the purpose of explaining the apparent discrepancy. He supposed the centripetal force to be not as $\frac{1}{d^2}$ but as $\frac{1}{d^2} + \frac{1}{d^4}$. In a very short time, however, he candidly gave up this theory, and announced the important fact that he had found the whole error to arise from his having in his approximation neglected some quantities as extremely minute, and supposing they could not affect the result, whereas one of the quantities upon which the result mainly depends, having a small numerator, is nearly doubled by the introduction of the quantities omitted. Upon again going through the investigation without the omissions, this great geometrician had the satisfaction of finding that the result made the motion of the moon's apsides agree with the fact; and both Euler and D'Alembert now found that in their solutions they had fallen into the same error. Laplace has since in his great work* given a complete investigation of the problem, and the results to which he is conducted by the theory are also

* Méc. Cél. liv. vii. s. 16.

most satisfactory. He finds the amount to differ only one four hundred and forty-fourth part from that given by observation, which reduced to our sexagesimal degrees, is only a difference of 24″ 12‴ from the observed amount. His solution in the case of the nodes does not come so near the observation; it is only correct within the 350th part; and yet the success of the theory in the case of the nodes was always reckoned its great victory in the hands of its author, while the case of the apogee cast some doubt upon it. Laplace made a discovery in the course of this inquiry of a similar variation in the apogeal movement, and that it becomes slower at the rate of 15″ in 100 years, which the recent observations confirm.

It was certainly impossible for the Newtonian theory to obtain a more brilliant triumph.* But it deserves to be mentioned, that the statement made by Bailly is even more incorrect upon this

* For Clairaut's papers, see Mém. de l'Acad. des Sciences, 1745 and 1748. But there is an admirable paper of the same illustrious mathematician on the motions of the orbits in the Mém. for 1754. The first cited volume contains both Clairaut and D'Alembert's famous investigation of the problem of the three bodies to which reference is made in the text as having been undertaken by them and Euler at the same time.

subject of the moon's apsides than upon the motion of the planetary axis. He asserts that Newton represented the theory as "giving the quantity of the moon's apogeal motion with exactness;" and that this having been a mere dictum of his without a demonstration, philosophers waited to find it proved by subsequent inquiry, as the theory had been on so many other points. The great inaccuracy of the substance is assuredly not rendered the less distasteful by the manner of this statement. "Il avait souvent parlé à la manière des prophètes qui disent ce qu'on ne peut voir: alors c'est la foi qui croit, il faut que la raison se soumette." (*Hist. de l'Astron.* iii. 150.) Newton never asserts anything which may not, from what he himself lays down, be strictly demonstrated. He certainly leaves much to be supplied; but he never leaves the reader who would, with due knowledge of the mathematics, follow his reasoning, to trust his word. Even the scholium at the close of the Lunar theory (after *Proposition* xxxv. *B.* iii.) where more of the investigation is omitted than perhaps in all the rest of the Principia together, may be followed argumentatively by a learned and diligent reader, as the Jesuits have shown in their inimitable commentary upon it. But touching the particular instance referred to by Bailly, nothing can be more contrary to the fact than his statement. Sir Isaac Newton in the general proposition which we analyzed above, after finding that any body acted upon by a disturbing force in the given proportion to the centripetal, will have by the theory a progressive motion of its apsides equal to 1° 31′ 28″, although he had not in the whole corollary made any particular application to the moon's motion, adds, "the apsis of the moon has a velocity twice as great nearly" (apsis lunæ est duplo veloci circiter), (*Cor.* 2. to *Prop.* xlv. *B.* i.);* and though in the proposition in which he applies his theory to find the disturbing force of the sun, (xxv. *B.* iii.) he finds it to be to centripetal force as 1 to 178½ nearly (or double what he had argued upon in the former proposition), he is so far from deducing from thence any inference that the apsides by the theory move 3° in each revolution that he makes no application at all of the proposition to finding their motion; but in the celebrated scholium where he sums up all the disturbances, in

* It is remarkable that these words are not in the first edition of the Principia.

treating of this motion, he expressly shows that it only comes out to be anything like the true motion of 3° by an assumption contrary to the theory; that is, by taking not the true equation to the sun's mean motion, but the equation on the hypothesis of its following the inverse triplicate ratio. The words above quoted from the general proposition upon the apsides in the first book, are quite sufficient to protect Newton's memory from any such aspersion as that now under consideration.

It may further be remarked, that Bailly's general criticism on Newton's whole investigation of the moon's motion is singularly unfortunate. He represents him as having only given a rough sketch of the subject which he left others to fill up; and he says, that this is the part of Newton's work most involved in obscurity; that, concealing the route he pursued, he plainly has not taken the problem in its full extent, but only shown generally, and by a few examples, that those irregularities could be deduced from the theory; though he renders ample justice to Newton's transcendent merits in other respects. But here Bailly has a far higher authority than his own against him, justly as his own name is held in respect. Laplace, who in his earlier works* had seemed not sufficiently impressed with the inestimable value of that part of the Principia, and had, while he distinctly gave the work at large the "pre-eminence over all other productions of the human understanding," appeared to regard the theory of disturbed planetary motion, and especially of the moon's motion, as a sketch left for others to fill up when the calculus should be more improved, in his last work, the concluding part of the Mécanique Céleste, published the year before his death, distinctly declares this very portion of the Principia to be among the greatest, if not the greatest, monument of the author's genius. "Je n'hésite point à les regarder (recherches sur la théorie de la lune) comme une des parties les plus profondes de cet admirable ouvrage."†

It remains, however, that we mention an unaccountable statement of the truly great geometrician whom we have last cited. In treating of the history of the lunar theory, he says that Newton, when seeking the correction of the sun's disturbance of the moon's gravitation towards the earth, "supposes that disturbance to be $\frac{2}{357}$ of the moon's

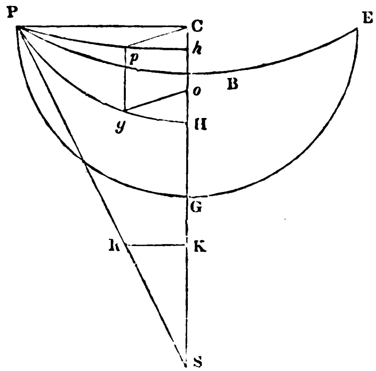
* Système du Monde, liv. v. chap. 5.

† Méc. Cél. liv. xvi. chap. 1; published in 1825.

gravity, or that which results from the observed amount of the lunar apogee." (*Méc. Cél. lib.* xv. *chap.* 1.) For this he refers to Book iii. Prop. iv of the Principia, which is evidently a wrong reference, that proposition, and indeed that part of the book, treating of other subjects. Nor can any place be found which Laplace could have had in his view, except the Twenty-first proposition of the Third book, in which the sun's disturbing force on the moon's motion is investigated. But respecting that proposition, it is wholly inaccurate to say that he there makes any hypothesis or assumption of the proportion between the disturbing force and the moon's gravity; for he deduces the proportion of 1 to $178\frac{29}{40}$, (or which is nearly the same thing, 2 to 357,) from the duplicate ratio of the periodic times, and deduces it as a consequence of the Seventeenth corollary to the Sixty-sixth proposition of the First book, which corollary comes easily from the Second corollary of the Fourth proposition of the First book. It is, therefore, wholly impossible to represent that position as a mere assumption to suit the observation of the moon's actual variations.

ii. The next subject of consideration is the motion of bodies along given surfaces, not in planes passing through the centre of forces, to which case our inquiries have hitherto been confined.

Let a body move in any plane in a trajectory, by a force directed towards a centre out of that plane, and we are to examine its motion under two heads, as we did the motion of a body when the centre was in the plane of the trajectory; that is, first, the curve described by the given force, and next the force, with the velocity, when the curve is given.

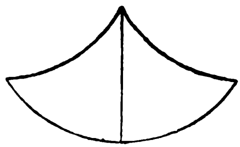


For this purpose, call the perpendicular S C to the plane from S the given centre, p, this being the shortest line from the point to the plane; the distance S P from the centre to any point P of the curve D; the distance C P $= d$ of that point P to the centre in the plane, that is, to the point C where p falls on the plane; and let F, the central force, be represented by R S. It is evident that the force R S, acting in the line P S (without the plane), is compounded of two, R K and S K, of which R K only can have any effect on the motion in the plane, the other S K which tends to draw the body out of the plane being by the supposition nothing, because the body moves wholly in the plane C P B E. But by similar triangles, R K $= \frac{SR \times CP}{PS} = \frac{F.\ d}{D}$; therefore if the proportion of the centripetal force to the distance be known, that is, if $F = D^n$, R K $= d.\,D^{n-1}$. But $D^2 = d^2 + p^2$ and $D = \sqrt{d^2 + p^2}$; therefore R K, the force acting at P towards the centre C, is $d \times (d^2 + p^2)^{\frac{n-1}{2}}$, which gives it in terms of the distance C P, and the given line S C. Thus if the central force is as the distance S P, the force acting towards the centre becomes equal to d, or as the distance on the plane. So if the central force is inversely as the distance, then $n = -2$, and the force to the centre on the plane is $\frac{d}{D^2}$ or $\frac{d}{d^2 + p^2}$, and if it is inversely as the square of the distance, the force on the plane is

$\frac{d}{(d^2 + p^2)^{\frac{3}{2}}}$. But the central force being given in the plane, the investigation is reduced to that formerly explained for finding motions and trajectories when the centre is in the same plane with the motion. Hence in the case first put, of the force towards S being as D, and the force towards C being, consequently, as *d*, it follows from what was formerly shown respecting motion in the same plane, that the curve described on the plane of the centre C, or P B, in this case is an ellipse; that the times in which the ellipse is described will be the same in whatever plane the bodies move; and that if the ellipse, by lengthening its axis indefinitely, becomes a straight line, the vibrations of the body in that line will be performed in equal times to and from the centre on both sides of it.

By a somewhat similar process we find the motion and trajectory of a body moving on a curve surface by a force directed towards a given centre in the axis of the solid of revolution which forms that curve surface. It is first shown, that if from any point of the trajectory P *g* H on the curve surface (which being a curve of double curvature we shall call the double curve), a perpendicular *g o* be drawn to the axis C S, and from any other point of the axis there be drawn a line equal and parallel to *g o*, as C *p*, C *p*, will describe areas proportional to the times. By means of this proposition and the former ones respecting motion in the same plane, we are enabled to find the curve P *p h* on the plane P B E, the points of which curve are, as it were, a projection of the trajectory, or double curve, P *g* H on that plane; and having found P *p h*, the double curve is found by drawing perpendiculars to the plane P B E, from the curve P *p h* to the curve surface P G E, whose form is given. Thus suppose the solid to be a cylinder, in which case the curve P *p h* will be the circle which is the section of the cylinder; then if the central force acts (by S being removed to an infinite distance) in lines parallel to the axis, and we suppose the body to begin its motion in the double curve P *g* H, with the same velocity as that given or central velocity, with which it would describe P *p h*, the double curve is found by taking the ordinates *p g* in a given proportion to the square of the circular arch P *p*, or as $\frac{Pp^2}{m}$; and consequently P *g* H is a species of quadratrix described on a cylinder.

The motion of pendulums is evidently a case of motion in a curve surface by a force directed towards a point in the axis of the solid, of which solid the curve described by the pendulous body is a section; and Sir Isaac Newton discusses this subject fully. As subservient to this inquiry, he gives some important properties of the cycloid, or rather of the hypocycloid and hypercycloid, for he is not satisfied with the investigation, which is sufficiently easy, of the ordinary cycloid's properties, the curve described by a point in a circle or wheel running along a straight line, but examines what is more difficult, the properties of the hypercycloid and hypocycloid, or the curves described by a wheel moving on the convex, and the concave great circle of a sphere respectively. Of these properties the most important is this. If D be the diameter of the sphere, and d that of the wheel, the length of the hypercycloid is equal to four times $\frac{d}{D} \times (D + d)$, or four times the length of a fourth proportional to the sum of the two diameters, the wheel's diameter and the sphere's. It is then shown how a pendulum may be made to vibrate in a given cycloid, or rather hypocycloid, namely, by taking a distance, which is a third proportional to the part of D, which the hypocycloid cuts off (that is, the distance of the

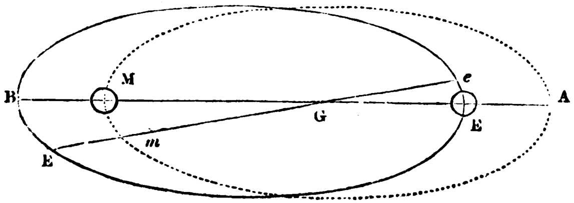


hypercycloid from the centre of the sphere) and $\frac{D}{2}$; and from that distance S so found, drawing two cycloids touching the sphere, or its great circle, and meeting in the point so found. If to that point S, a flexible line or thread be attached and bent round one of the cycloids S P, it will unrol itself and then bind itself round the other cycloid S P′, and its extremity will describe the cycloidal curve P P′ required, one of whose properties is, that all the vibrations in its arches are performed in equal times, however unequal the lengths of these arcs may be, provided that the centripetal force is in each part of the curve directly as the distance from the centre, and that no other force acts on the moving body.

But the same solution may be generalized and applied to any given curve whatever; for the curves found and along which the flexible line is traced and from which it is then unrolled, are the evolutes of the given curve, and are found in each case by means of the radius of curvature, being the curve formed by its extremity, or the locus of the centres of the osculating circles to all the given curve's points. If the curve in which the body is to move be a circle, the evolute is, of course, a point, the centre of that circle, the radius of curvature being that of the circle. If the curve is a conic parabola, it will be found that the evolutes, or the lines from which the pendulum's thread must wind off, are cubic parabolas, whose equation is $y^2 = (\frac{2}{3}x)^3$, the length of the pendulum being unity. The only case of the problem investigated by Sir Isaac Newton is that of the cycloid, which has the remarkable property, that its evolute is an equal and similar cycloid, a property which it has in common with another curve, the logarithmic spiral, whose tangent makes with the radius vector a constant angle. He investigated the case of hypocycloids and hypercycloids, rather than the common cycloids, because it is that of the earth's gravity, which above the surface decreases inversely as the square of the distance from the centre, but within the sphere increases as the distance simply.

It follows, from the propositions respecting the vibration of pendulums, that the times of the descent of falling bodies may be compared together and with the times of vibrations of the pendulum, so that the time of a vibration round a given centre being given, as a second, the time of the falling body's descent to the centre of forces can be found, or the equal time of vibration in the circular arch of 90° with any radius. The time is to the given time as 1 to $\sqrt{\frac{L}{D}}$, L being the length of the pendulum and D the distance from the point of suspension to the centre of forces; and since D becomes infinite and the lines in which the central force acts parallel, and since half the length of the pendulum is to the line fallen through in the time of one vibration as 1 to 9,869 nearly (the proportion of the square of the diameter to that of the circumference), we can easily ascertain the force of gravity at any point by the length of the pendulum vibrating seconds. It is found to be in these latitudes about 34·44, consequently a body falls in a second through about 16 feet 9 inches.

Hitherto we have only considered the motions and trajectories of bodies acted upon by forces directed towards a fixed centre whether in the plane of their motion or out of that plane, and supposing that plane either to be fixed or to be moved round the centre of forces. But as action and reaction are equal and opposite, by the third law of motion originally stated, it is evident that the case of a fixed centre cannot exist when the attraction, which we call the centripetal force, proceeds from a body placed in the centre, unless, indeed, some counteracting force shall fix this body to one point; for if no force exists but the mutual action of the two bodies, the central one must be acted upon by the one which moves round it, and its position must be affected by this action. Hence, for example, if there were only two heavenly bodies, M and E, and the one, M, moved round the other, E, by a projectile force originally impressed upon it, the other, E, would also move round M, unless the mass of the latter body was infinitely small, and its attraction, proportional to this mass, could not sensibly affect the larger body. Again, if two bodies, the one moving round the other, both together move round a third, S, the action of this third will affect the motions of the other two relatively to each other. Thus each smaller system will be affected, both as to the motions and orbits of the bodies composing it, by the action of the body in the common centre of the whole, and they will also be affected by the action of the bodies in the other systems, having the same common centre. The inquiry, therefore, divides itself into two branches; *first*, the difference between the motions which we have hitherto been considering when the centre was fixed, and the actual motions of the system, as that of the moon and earth revolving round each other with a moveable centre; *secondly*, the still more important difference between the motions already considered, and the actual motions, which difference is caused by the mutual actions of the whole bodies on each, and varies both the motions and the orbits of all.



i. Suppose two bodies mutually attracting each other and impelled by a single original force of projection, as E and M, their centre of gravity being G; it is clear that if M moves a very small space to *m* by the attraction of E, so will E move to *e* by the attraction of M, and the two triangles E G e and M G m will be similar in all respects; for the lines M G, m G and E G, e G are proportional, because the segments of the lines E M and e m are always in the same proportion, G being the centre of gravity, and those segments, therefore, inversely as the masses of E and M. Therefore the curves which the bodies describe round the centre of gravity will be entirely similar. In like manner they will describe similar curves each round the other, and the radius vector of each from the other, as well as from the centre of gravity will describe areas proportional to the times. It follows from this and from what was before shown respecting centripetal forces, that the two bodies will move in concentric ellipses round one another and round their common centre of gravity, if the centripetal force is as the distance, and that each will describe one or other of the conic sections, having the other, or the common centre of gravity, in the focus, if the centripetal force is inversely as the square of the distance. In like manner, because of the ratio between the squares of the periodic times and cubes of distances, it may be shown that if T be the periodic time of the bodies moving round their centres of gravity and t the periodic time of

M moving in a similar figure round E at rest, $T : t ::$ $\sqrt{E} : \sqrt{M + E}$; and if these bodies move with forces inversely as the squares of the distances round their centres of gravity, if A be the greater axis of the ellipse described by M round E, and a the greater axis of the ellipse it would describe round E at rest in the same time, and if $M + E : m :: n : E$, then $A : a :: M + E : m$. Hence, if we have the periodic times of the planets, we can find the greater axes of their orbits by taking A^3 to a^3 in the proportion of T^2 to t^2 (the ellipse being supposed described round the sun), and multiplying it by $\frac{n}{E}$. So the masses may, likewise, be found from the distances.

The motions and paths of bodies thus mutually acting are now to be considered, and first our author shows, that if two bodies act on each other, and move without any other, or foreign, influence whatever, their motion will be the same as if, instead of acting on one another, some third body placed in the centre of gravity acted upon each of them with the same force with which each acts on the other; and the same law will prevail (but referred to the distances from the centre) which prevailed in their mutual actions when referred to their distances from each other. Suppose the bodies M and E to attract with forces directly as their masses M and E, and inversely as any power n of their distances, that is their attraction to be as $\frac{M}{D^n}$, and $\frac{E}{D^n}$, and that the distances of the centre from M and E are C and c respectively; then because $C : c :: E : M$, and $C : C + c$ (or D) $:: E : E + M$, a body in the centre will attract M with a force as $\frac{E}{D^n}$, if it be equal to $\frac{E \times C^n}{D^n}$, that is equal to $\frac{E^{n+1}}{(E + M)^n}$, and, in like manner, it will attract E with a force equal to $\frac{M}{D^n}$, if it be equal to $\frac{M^{n+1}}{(E + M)^n}$.

If n is 2, or the force be as the inverse square of the distance, the body placed in the centre will be equal to $\frac{M^3}{(E + M)^2}$; if $n = -1$, or the attraction be directly as the distance, the body will be in both the case of M and E equal to $E + M$; and if the attraction be as the square of the distance directly, the central body will be in the two cases of the two bodies $\frac{(M + E)^2}{M}$ and $\frac{(M + E)^2}{E}$ respectively.

Next as to the absolute trajectory of the bodies thus acting on one another, or their path in space, we have an investigation analogous to those inquiries formerly instituted where the centre of forces was fixed. For the body or bodies being known (by what we have last shown) whose mass gives at the centre the same attractions as the two bodies exercise on each other, we can determine for each of these bodies the path in which it will move, provided we know the initial direction and velocity. Thus let $m = 2$ in the last expressions, we have for the mass by which M is attracted towards the common centre of gravity, $\frac{E^3}{(M + E)^2}$; and proceeding as was formerly shown in the case of immoveable centres, we find that if the curve described round the centre at rest be a circle, if that centre moves in a straight line, the orbit in space will be a cycloid; if the centre moves in a circle, it will be an epicycloid or hypercycloid, and if the curve be a conic parabola, the motion of the centre will change this into a cubic parabola, which will thus be the path arising from its parabolic motion combined with the advance of the centre of gravity. The moon in this way describes thirteen cycloidal curves in a year, and all of them concave towards the sun.

It appears then, that the orbits of the system composed of our earth and its satellite, must be considered as traced, not by either of these bodies but by their centres of gravity. While neither body describes an ellipse round the sun, but both revolve round each other and round their centre of gravity, the centre itself describes an elliptical line, a line which would be a perfect ellipse if no disturbances of another kind than these which we have been considering interfered to alter the form of the orbit. To these disturbances we now proceed.

ii. While the primary planets and their satellites are influencing each other, and while the whole motion of each subordinate system round the sun, the common centre, is the elliptical orbit described by the centre of gravity of each such system, there are disturbing forces exerted on each planet by the rest, and on the motions of satellites by the action of the sun also; so that many sensible deviations take place from the motions, and from the orbits, which those bodies, both primary planets and satellites, would have, if they moved round the common centre undisturbed; that is, if they described elliptical orbits round the sun by his attraction, without any other force acting on them, except that attraction of the sun on each planet, and the attraction of each planet on its satellites. If no such disturbances existed, and the only forces that acted were the mutual actions of the primary and satellites on each other, and of the sun on the common centre of gravity of the primary and satellites, the centre would describe an ellipse round the sun, and the primary and satellites would describe ellipses round each other and round that centre of gravity. This, however, is not the case; and we are now to consider the effects of the disturbance occasioned by the sun's action upon the satellites, and the disturbance occasioned by the action of the planets on one another. This forms the subject of Sir Isaac Newton's investigations in the second branch of that section which we have been considering; an inquiry regarded by some as the most extraordinary portion of the great work which forms the principal monument of his genius. From this opinion it is difficult to withhold our concurrence; but it may be admitted that here, as in the operations for finding orbits from given forces and conversely, the great improvements of modern analysis have afforded easier and more manageable methods of investigation. That this must be true as regards the planetary disturbances, will be apparent upon a little reflection.

The grand problem in every case is to find the precise effect of a disturbing force upon the path of a given body moving by a combined centripetal and projectile force; and what has been called the Problem of Three Bodies presents the simplest case of the question, being the determination of the motions of two bodies acted upon by one another and by a third body. But though this is the simplest case of the general question, it has been found to present difficulties of the highest order, and a general and rigorous solution of it has been found to exceed the powers of the most improved analysis. In the time of Sir Isaac Newton, that analysis of which he was the inventor had not attained any thing like its greatest perfection. Hence, in grappling with the subject, he had much of the difficulty to contend with, which made him give less convenient formulæ than we now possess for the solution of the other problems relating to orbits and motions. The mere improvement of the integral calculus by the advantageous approximations through series, logarithms, and the arithmetic of sines, would have afforded important facilities for these inquiries; because the solution must come always to an integration. Accordingly Euler, D'Alembert, and Clairaut, availed themselves of that improvement to investigate the problem, as we have already seen. But soon after their researches had led to the important result formerly described, a great refinement was introduced into the calculus, which bore directly upon the subject of these inquiries; and this exceedingly facilitated the solution of the problem in its more extended application. We allude to the invention of the Calculation of Variations by Euler and Lagrange.

We have in the introductory part of this Analytical View explained that this calculus enables us to examine the transition of one curve into another in certain circumstances, by showing how those lines may be found which have certain properties in relation to other lines of a different kind, and thus to investigate problems with respect to curves, whose nature changes under the investigation, because the relation between their co-ordinates is variable, and is indeed the thing sought for. It is evident, therefore, that this calculus has its immediate application to the subject in question. For the effect of the disturbing force is to change at each moment the nature of the path, which, but for that force, would be described; or the inclination of orbits to one another, which, but for such disturbances, would subsist; or the position in space, which, but for the disturbance, these orbits would have. Now, those changes produced by mutual disturbances, really comprise all the effects of the disturbances on the planetary system. Thus, beside the precession of the equinoxes and the motion of the apsides and nodes, which we have just now generally stated, the alteration in the form of the curve includes also the change of its eccentricity, and the acceleration or retardation of the motion itself. Hence, we have at once proved that the determination of those effects which arise from disturbing forces, is in a peculiar manner the province of this new and refined analysis, the Calculus of Variations. Therefore, beside the facilities afforded by the improvement of older methods of investigation, the addition of this new instrument to our means of solving the problem has established an entirely novel method, and opened an almost unknown field of inquiry, from which the original author of all these discoveries was necessarily shut out. Instead, therefore, of minutely going over the steps of his solution, as applied to the celestial motions; we shall show the course which he pursued by demonstrating its fundamental principles; but we shall begin by stating concisely the results of the more recent investigations as affecting the science of physical astronomy; and shall reserve the fuller discussion of this subject for the account of Laplace's work.

In considering the motions of the planets and their satellites round the sun, we may first regard him as from his magnitude and distance so little affected by their attractions, that his motion is trifling, and cannot sensibly affect that of the other bodies; so that he may be viewed as at rest; and then the smaller bodies will both move round one another, and round the larger and more distant body as if he were fixed. But the movement of these bodies will not only be thus affected by their mutual actions; they will be affected in their motions round one another by the action of the third body, the sun; and this action will disturb and alter their relative motions, as regards both their velocities, the forms, and the positions of their orbits. Thus the position of the moon's path round the earth, is affected by the sun's attraction, that is, by her gravitation towards the sun, which combines with her gravitation towards the earth to determine her absolute motion; and both the position of the axis of her orbit (the line of apsides), and the position of the line joining the intersections of her orbit's plane with the plane of the earth's orbit (the line of nodes), are continually changing, and we have seen in a particular manner how the apsides revolve in one period of time (about nine years), and the nodes in another, (about nineteen years.)

But there is a variation in the rate at which both the line of the apsides and the line of the nodes revolve. The quantity by which both of these lines advances in each year sensibly decreases; so that the period in which each effects a complete revolution becomes longer and longer. It appears that the former line revolves now $8' \, 2''$ slower than in the earliest ages of astronomical observation, about $25\frac{1}{2}$ centuries ago; the latter line only $1' \, 42'' \, 14'''$; the former motion diminishing each century by $36'' \, 41'''$; the latter by $7'' \, 51'''$.

It is equally found that the disturbing forces accelerate the moon's motion by a very small quantity; or that she revolves round the earth in a period of about 11″ 7‴ shorter than she did a century ago; her angular velocity being increased between the 12 and 13,000 millionth part of her total velocity in the period of 100 years; making the yearly acceleration wholly insensible; and the total acceleration, or shortening of her periodic time since the creation of our species 60 centuries ago, only 11 minutes and 7 seconds, supposing it to go on as the times, but as it increases in a lower proportion (probably as the cube of the times) its total amount is more considerable, and Laplace reckons it at about 7′ 30″ for the last 25½ centuries. This acceleration had not been unobserved in Halley's time, and it was discussed acutely by Mayer; but its cause was first discovered by Laplace; it is the sun's action upon the moon, combined with the variation in the orbit of the earth, the eccentricity of which has been diminishing regularly, though by an extremely small quantity (only $\sqrt{.0000007667}$ of the greater axis of our orbit), so that the orbit has been slowly approaching more and more to the circular form. It is a great proof of the usefulness of the calculus in these investigations that this great geometrician appears to have discovered the connexion between the earth's diminishing eccentricity and the acceleration of the moon's mean motion, by the careful examination of the mere equation or algebraical expression. For the reciprocal of the semi-axis of the moon's orbit $\frac{1}{a}$, as influenced by the sun's attraction combined with the earth's, is found to be represented by an expression, which, among other terms, contains this $-\frac{3}{4}\ \frac{a^2 m' e'^2}{a'^3}$ in which a' is the semi-axis of the earth's orbit, m' the mass of the sun, and e' the eccentricity of the earth's orbit. Consequently, as e' decreases, $\frac{1}{a}$ increases, the term being negative; and therefore a itself decreases as e' decreases; in other words, the moon's orbit is diminished, and her velocity augmented, in consequence of the earth's eccentricity decreasing. But if the diminution of the greater axis is not admitted as necessarily lessening the orbit, we may recollect the relation between the times and the mean distances, the former being as the cubes of the latter; and the mean motion is, of course, inversely as the periodic time. However Laplace furnishes us with a still closer reason, and illustrates the use of the calculus, as it were, by a new triumph, in another part of the *Mécanique Céleste.** For the equation of the mean angular motion is shown to be $n = \frac{t\sqrt{\mu}}{a^{\frac{3}{2}}}$, t being the time, a the transverse axis, and μ, the sum of the masses of the two bodies, in this case the moon and the earth. Therefore n, the mean motion, must necessarily be accelerated as a, the axis, is diminished. And here in passing, we also observe how Kepler's law of the sesquiplicate ratio may be proved, but only if we make $\mu = \mathrm{S}$ (the sun); and neglect the mass of the planet. For take two planets whose mean motions are n and n' round a third body, and their mean motions being as $\frac{t\sqrt{\mu}}{a^{\frac{3}{2}}}$ and $\frac{t'\sqrt{\mu'}}{a'^{\frac{3}{2}}}$, and because ($2\pi$ being 360°), $n\,t = 2\pi$, therefore $t = \frac{2\pi}{n}$, and $t' = \frac{2\pi}{n'}$, or $t = \frac{2\pi . a^{\frac{3}{2}}}{\sqrt{\mu}}$, and $t' = \frac{2\pi\, a'^{\frac{3}{2}}}{\sqrt{\mu'}}$, consequently $t^2 : t'^2 :: a^3 : a'^3$, being Kepler's law, which is thus demonstrated. But it is only demonstrated and is only true if $\sqrt{\mu}$

* Liv. ii. ch. 3.

is the same to both planets, that is, if $\mu = S$ in each case. Now, this may be assumed in the case of those bodies revolving round the sun, or of the satellites of Jupiter and Saturn revolving round those primary planets, because of the great disproportion between the central body and the others, (the largest of them, Jupiter, being less than a thousandth part of the sun.) But the law would not hold true if μ were taken, as in strictness it ought, as $S + P$, the sum of the masses of the central and the revolving body; for then μ would differ in each instance, and the sesquiplicate proportion would be destroyed. Hence, we arrive through the calculus at this important conclusion, that the law only holds, if the mutual actions of the planets on each other are neglected; and that, therefore, the law is not rigorously true where, as in the case of the earth and others, the actions of the other planets are sensible.

Again, the inspection of the algebraical expressions shows that the variation in the eccentricity of the earth's orbit produces, likewise, the retardation of the apsides and nodes; and this discovery was also made, apparently, by the mere inspection of the expressions which the calculus had furnished.

Thus the expression for the motion of the perigee (or apsides) involves the integral $\int e'^2\, d\, v$ (v being the true anomaly),* and this quantity is positive. Therefore the decrease of the eccentricity of the earth's orbit, causes a decrease, also, of the perigeal motion of the moon. And one of the terms of the equation to the motion of the nodes contains the same integral $\int e'^2\, d\, v$; consequently the same eccentricity is likewise the cause of the variation in the period of their revolution.†

Now we have seen how extremely small these irregularities in the moon's motion are which the theory gives by this analytical process, and that they are hardly sensible in a whole century; yet it is found that the deductions of the calculus are in a remarkable manner confirmed by actual observation. Practical astronomers, for example, wholly ignorant of Laplace's discoveries, have ascertained that the secular variation in the motion of the moon's apsides, ascertained by comparing the eclipses in the Greek, Arabian, and Chaldean astronomy, with

* Angle of the radius vector with the axis of the orbit.

† Méc. Cél. liv. vii. ch. 1. This wonderful chapter is a mere series of integrations, and contains, from the inspection of the equations, those singular discoveries respecting the laws of the universe.

those of the last century, is about 3. 3, or 33 tenths of the moon's mean motion, and this is the exact result of the calculus. Laplace also discovered, chiefly by similar means, a very small secular inequality in the moon's motion never before suspected, and produced by the sun's attraction.* It was found by observing that the divisor of some of the fractional terms of the equation which shows the inequality is extremely small, and that, consequently, the irregularity may become sensible. A correction of the tables was thus introduced by this great geometrician, in which the theory approaches, on an average, to within $\frac{1}{800}$ of the actual observation. The sign of this inequality being negative, it is a retardation of the mean motion, and is to be set against the secular acceleration. It must be observed, moreover, that the errors of the theory, as compared with the observation, are half of them by excess and half by defect; so that they may be said to balance each other. The maximum of this inequality is little more than 15″, and its period is 184 years.

Hitherto of the moon; but we are, in like manner, conducted by the same refined, though complicated, analysis to the variations in the orbits,

* Méc. Cél. liv. vii. ch. 5.

and consequently in the motions of the earth and of the other planets, as well as of the satellites of Jupiter and of Saturn. The most remarkable variations produced upon these orbits are the changes in their eccentricity and in their aphelion; the former being constantly, though slowly, shortened, the latter moving round by slow revolutions, as the line of the moon's apsides revolves, but revolves much more swiftly.

The expressions obtained in the case of any one planet for the eccentricity and perihelion longitude (revolving motion of the axis), are mainly composed of the masses, distances, eccentricities, and perihelion longitude of the disturbing bodies with the known eccentricity and longitude of the planet in question at a given epoch. Hence we perceive that on these circumstances depends the variation of the eccentricity and the revolution of the axis of the planet. Thus the secular variation of the eccentricity of the earth's orbit is 0.000045572 of *e*, the eccentricity which at the epoch (1750) was 0.016814 of the semi-axis major of its orbit, and it has the negative sine in the expression; consequently the eccentricity is on the decrease, as we before observed. This diminution of the eccentricity amounts to about $18'' \, 79'''$ yearly (or about 3900 miles). We have already observed that the annual revolving motion of the axis of the earth's orbit is 11″ 53‴, and its period 109,060 years. The examination of the expressions for these irregularities shows, as might be expected, that Mars, Venus, and Jupiter bear the most considerable share in producing the variations.* But it is a truly remarkable circumstance that the direct action of those planets upon the moon's motion is hardly sensible compared with their indirect, or, as it is sometimes called, reflected action upon the same body, through the medium of the sun and the earth. For these planets, Mars, Venus, and Jupiter, by altering the eccentricity of the earth's orbit, very sensibly affect the motions of the moon, as we have seen, while directly their action is incomparably less perceptible.

The perihelion longitudes of all the other planets are increasing, or their orbits advancing, except Venus, whose apsides are retrograde; and the eccentricities of Venus, Saturn, and Uranus, are decreasing, like that of the earth, whilst those of the other planets are on the increase. These variations are greater in Saturn than in any of the others,—considerably greater than the variations of Mars,

* Méc. Cél. liv. ii. ch. 6, 7, 8; liv. vi. ch. 7.

which comes the nearest to them. The variation in the eccentricity of Jupiter's orbit is nearly three times as great as in the Earth's; that of Saturn between five and six times greater than the Earth's; while the variation in the perihelion longitude of the former is about five-ninths of the Earth's variation; and Saturn exceeds the Earth's in the ratio of about 25 to 18, and exceeds that of Mars only somewhat more than as 49 to 48.

When the attention of mathematicians and astronomers was first directed closely to examine the disturbances of these planets, it appeared hardly possible to reconcile such vast and numerous irregularities, as were found to exist, with the theory of gravitation, or indeed to reduce them under any fixed rule whatever. The case seemed to become the more hopeless when so consummate an analyst as Euler, the great improver of the calculus, failed in repeated attempts at investigating the subject, committing several important errors which for a time were not detected, but which showed, or seemed to show, a wide discrepancy between the theory and the observations. By one discovery, indeed, to which his researches led him, he may be said to have laid the foundation of the most extraordinary step which has been made in the knowledge of the planetary system. We allude to his theorem on the periodicity of the eccentricities and aphelia of Jupiter and Saturn. But in most other respects his attempts signally failed. D'Alembert made little progress in this inquiry; but at length Lagrange, and still more Laplace, by applying all the resources of the calculus, in its last stage of improvement and after the method of variations had been systematized, succeeded in reducing the whole to order, and discovered, while investigating these motions, the great law of the stability of the universe.

The circumstance which mainly contributes to render the irregularities in the motions of two planets great, and which especially augments the disturbance of Jupiter's satellites, is that the mean motions are commensurable after a very remarkable manner, Five times the mean motion of Saturn are equal to nearly twice that of Jupiter: and the three first satellites of Jupiter are so related to each other, that the mean motion of the first, added to twice the mean motion of the third, is equal to three times that of the second; while the longitude of the first added to twice that of the third, and subtracted from three times that of the second, makes up exactly 180°. Laplace showed, that this proportion of it was not originally fixed between those satellites, and must have been established by the action of the attractive and disturbing forces,* and it is a truly remarkable thing, that when the theory had given a value for the three mean motions $M - 3m + 2\mu = 0$, the comparison of the eclipses for a century was found to make the expression only 9″, and consequently to tally with the theory within that very small difference. The observation of the effects which were produced upon the equations which resulted from the analysis, by the proportions above stated between the mean motions of Jupiter and Saturn, induced Laplace to suspect that this made quantities become of importance, which from the high powers of the denominators might otherwise have been insignificant. For one of the terms in the expression by which δr (variation of the radius vector of the first satellite) for example had for its denominator $4(n' - n)^2 N^2$ in which n and n' are the mean motions of the first and second satellite, and N a

* Méc. Cél.. liv. vi ch. 1. 2. 12. 13; also for the analytical investigation, see liv. viii. throughout, and liv. ii. ch. 8. 865.

composite quantity not materially differing from n', which differs hardly at all from $\frac{n}{2}$, inasmuch as $n = 2n'$, while $n' = 2n''$ (n'' being the mean motion of the third satellite), and hence the above denominator becoming little or nothing, the term is of large amount, and so of δv, the variation of the anomaly.* He accordingly undertook the laborious task of examining this complicated subject by considering all these quantities; and he arrived at the discovery of, among other inequalities, a retardation of Saturn's motion of about 3″ 6‴ yearly, and an acceleration in Jupiter's motion of about 1″ 18‴. Another irregularity in Saturn's motion with respect to the vernal and autumnal equinox had been observed by astronomers in the last century, and could not be explained. Laplace found this, like all the rest, to follow from the Newtonian theory. In short, when summing up the subject in one of his concluding books, he naturally and justly exclaims, "Tel a été le sort de cette brillante découverte, que chaque difficulté qui s'est élevée, est devenue pour elle un nouveau

* Méc. Cél. liv. viii. ch. 1. 4.

sujet de triomphe; ce qui est le vrai caractère du vrai système de la nature."*

There is no sensible disturbanee produced by any of the satellites, except the moon, upon the motion of their primaries, from the extreme smallness of their masses compared with those of the sun and of their primaries; for δr is equal to a series in which $\frac{m}{M}$ $\frac{m^{i}}{M}$, $\frac{m^{iii}}{M}$, &c. are factors of each term,† m, m', &c. being the masses of the satellites, and M that of the planets, Now, in the case of Jupiter $\frac{m}{M} = \frac{1}{57710}$, $\frac{m^{ii}}{M}$ and $\frac{m^{iv}}{M}$ are somewhat greater, but the greatest of the four factors $\frac{m^{iii}}{M} = \frac{1}{11302}$ only. But in the case of the earth this factor amounts to about $\frac{1}{80}$, so that δr and δv become sensible; and will be so, even if, instead of $\frac{m}{M}$, we take the factor $\frac{m}{M + m}$, which is more correct.‡

* Méc. Cél. liv. xv. ch. 1.—Syst. du Monde, liv. v. ch. 3.

† Ibid. liv. vi. ch. 4.

‡ Ibid. liv. vi. ch. 10. 30.

When Laplace began his celebrated investigations of the orbits of Jupiter and Saturn, he found that on substituting numerical values for the quantities in the expression of the mean movement of the one body as influenced by the action of the other, the sums destroyed one another, and left the whole effect of this disturbing force equal to nothing, or the mean motion of neither planet at all affected by the other. The formulæ could be in each case reduced to terms only involving two coefficients, and these destroyed one another.* He soon found that the same principle applies to all the heavenly bodies; that their mean motions and mean distances (the great axes of their orbits) are not affected by any changes other than those which occur within limited periods of time; that consequently the length of the solar year is precisely the same at any one period of time, as it was at a period so far distant as to enable the changes which are produced within those moderate limits to be effected. This important proposition he demonstrated upon the supposition, that the squares of the masses, and the fourth powers of the eccen-

* Méc. Cél. liv. ii. ch. 7.; liv. xv. ch. 1.

tricities, and the angles of the orbits, are neglected in the calculus.* But Lagrange afterwards showed, that the theorem holds true, even if these quantities be taken into the account. The examination of the moon's motion demonstrates the same important fact, with respect to the permanency of the greater axes and mean motion of the planets; for if the solar day were now $\frac{1}{800}$ of a second longer than it was in the age of Hipparchus, the moon's secular equation would be augmented above 42 per cent., or would be in that large proportion greater than it now is known to be. Therefore there has not even been the smallest change of the mean movement of the planets.

The other changes which take place in the orbits and motions of the heavenly bodies were found by these great geometricians to follow a law of periodicity which secures the eternal stability of the system. The motion of the earth's orbit we have already seen is so slow, that its axis takes above 109,000 years to perform a complete revolution; but after that time it occupies precisely the same position in space as it did when this vast period of time began to run. So the eccentricity of the earth's orbit has been for ages slowly decreasing, and the

* Méc. Cél., liv. ii. ch. 7. and 8. (sects. 54 and 63.)

decrease will go on, or the orbit will approach nearer and nearer to a circle, until it reaches a limit which it never can pass. The eccentricity will then begin slowly to increase until it again reaches its greatest point, beyond which the orbit never can depart from the circular form. The same principle extends itself to all the planets. Thus the time of the secular variation of Jupiter's eccentricity is 70,400 years. All these deductions are the strict analytical consequences of the equations to the eccentricity of the planetary orbits obtained by the investigation of the total effect of the mutual actions of the heavenly bodies. There results from that analysis this remarkable theorem, that if the eccentricities of the different planets be called e, e', e'', &c., their masses m, m', m'', &c., and their transverse axes a, a', a'', &c., and the integration be made of the fluxional expression for the relation between the fluxions of the eccentricities multiplied by the sines of the longitude, and the fluxions of the time, and the relations between the fluxions of the eccentricities multiplied by the cosines of the longitudes and the fluxions of the time, $\left(\frac{d\,e \sin. \varpi}{d\,t}, \text{ and } \frac{d\,e \cos. \varpi}{d\,t}, \frac{d\,e' \sin. \varpi'}{d\,t, \&c.}, \text{ and } \frac{d\,e' \cos. \varpi'}{d\,t}\right)$

&c., we obtain the equation $e^2 . m . \sqrt{a} + e'^2 , m' . \sqrt{a'} + e''^2 , m'' \sqrt{a''}$, &c. $= C$* ; C being a constant quantity. Now, as all the motions are in the same direction, $\sqrt{a}$, $\sqrt{a'}$, &c., are all positive. Hence, it follows that each of the quantities $e . m . \sqrt{a}$, $e' . m' . \sqrt{a'}$, &c., is less than C; and suppose at any one period the whole eccentricities e, e', e'', &c., to be very small, which is known to be true, C, which at that period was the sum of their squares, must be very small, the other quantities m, m', &c., being wholly constant, and $\sqrt{a}$, $\sqrt{a'}$, &c., being invariable in considerable periods of time. Therefore, it is clear that the variation in any one of those eccentricities as e, never can exceed a very small quantity, namely, a quantity proportional to $\sqrt{C - e'^2 - e''^2}$, &c. The whole possible amount of the eccentricity is confined within very narrow limits. It never can for any body, whose eccentricity is e, exceed a quantity equal to

$$\sqrt{\frac{C - e'^2 . m' . \sqrt{a'} - e''^2 . m'' . \sqrt{a''} - \&c.}{\sqrt{m . \sqrt{a}}}}$$

Therefore the eccentricities never can exceed a very small quantity. Thus the changes which are con-

* Méc. Cél. liv. ii. ch. 6, 7 (sects. 53, 55. 57, 58).

stantly taking place in the planetary orbits are confined within narrow limits; and the other changes which are the consequences of this alteration of the orbits, as, for instance, the acceleration of the moon, which we before showed arose from the variation of the eccentricity of the earth's orbit, are equally confined within narrow limits. Those changes in the heavenly paths and motions oscillate, as it were, round a given medium point, from which they never depart on either hand, beyond a certain small distance: so that at the end of thousands of years the whole system in each separate case (each body having its own secular periods) returns to the exact position in which it was when these vast successions of ages began to roll. For similar theorems are deduced with respect to other revolutions of the system, whose general destiny is slow and constant change, but according to fixed rules, regulated in its rate, confined in its quantity, limited within bounds, and maintaining during countless ages the stability of the whole universe by appointed and immutable laws. Laplace examined in the last place the possible effects upon the celestial motions of the resistance of a subtle ethereal medium, and of the transmission of gravity or attraction not being instantaneous, but accomplished in a small period of time. The result of his analysis led him to disbelieve in both these disturbing causes. He found that in order to produce its known effects, the transmission of gravity, if effected in time, must be seven millions of times swifter than that of light, or 147 thousand millions of miles in a second.*

iii. The great system of most interesting truths which we have now been contemplating is the work of those who diligently studied the doctrines unfolded by Sir Isaac Newton, respecting the motions of bodies which act upon each other, while they are moving around common centres of attraction. He laid down the principles upon which the investigations were to be conducted; he showed how they must lead to a solution of the questions proposed, touching the operation of disturbing forces; and he exemplified the application of his methods by giving solutions of these questions in certain cases. Although his successors, treading in his steps, have reaped the great rewards of their learning and industry, and are well entitled to all praise for the skill with which they both worked and improved the machinery that he had put into their

* Méc. Cél. liv. vii., ch. 6; liv. x. ch. 7.

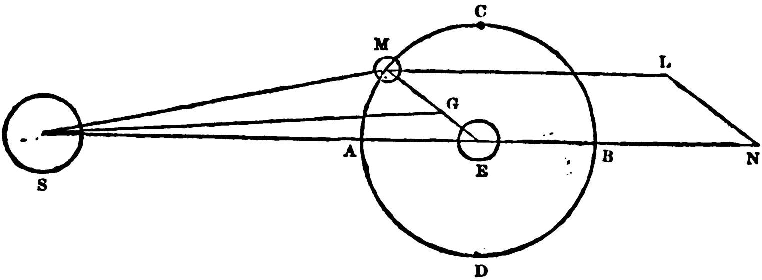
hands, at once improving the calculus invented by him, and felicitously applying it to advance and perfect his discoveries, the distance at which his fame leaves theirs is at least equal to that by which a Worcester and a Watt outstripped those who, in later times, have used their mechanism as the means of travelling on land and on water, in a way never foreseen by those great inventors. Strict justice requires that we should never lose sight of the truth repeatedly confessed by Euler, Clairaut, Delambre, Lagrange, Laplace, that all the advances made by them in the use of analysis, and in its application to physical astronomy, are but the consequences of the Newtonian discoveries, so that we are guilty of no exaggeration, if we regard the most brilliant achievements of these great men only as corollaries from the propositions of their illustrious master. Let us briefly see how he laid the deep and solid foundations of the fabric which we have been surveying.

After examining the motions of a system of two bodies with respect to one another, and their common centre of gravity, and in space, as those motions are affected by the mutual attractions of the two bodies themselves (in the manner which we have already described), Newton proceeds to the great problem of the three bodies, as it has been termed, because the solution is so difficult, that generally the attempt has been confined to the case of these only, this also being sufficient for determining the more important disturbances of the moon's motions. The inquiry, however, is general in the Principia, and its subject is, the motion, produced by the mutual actions of the bodies in a system upon one another. Thus, for example, the inquiry already analysed regards the effect produced upon their motion in space, by the mutual attractions of the earth and moon; that to which we now are proceeding regards their motion, as also influenced by the disturbing force of the sun, and indeed, even by the smaller but not evanescent forces of the other planets. So, as the former inquiry may be extended on the same principles to the motions of Jupiter and Saturn, and their satellites; this new inquiry applies also to the disturbances of their systems by ours, and of ours by them.

Newton begins by showing that if the attracting force increases, as the distance of the bodies from each other, any two, M and E, will revolve round their common centre of gravity G, in an ellipse having G for its centre. This is plain from what was formerly proved when treating of the conic sections, and also more lately respecting the centre of gravity. If, then, each of these is attracted by a third body S, in the same manner this force, being resolved into two, one parallel to the line joining M and E, the other parallel to the line joining E and G, the former force will only accelerate the motion of M and E round G by an addition to the mutual attraction of M and E, the latter will draw the centre G towards S or towards G′, the common centre of gravity of the three bodies, and combined with the action of M and E upon their centre G will make G revolve in an ellipse round G′, the common centre of the three, round which also, in like manner, S will describe an ellipse, G′ being the centre of those two ellipses. Thus the bodies M and E will describe an ellipse round the centre G, and the centre G and body S will describe ellipses round the centre G′, both G and G′ being the centres of these ellipses, and so of any greater number of bodies.—Moreover, the absolute amount of the attractive force in each centre will be as the distance of the centre from the bodies or centres of gravity severally, multiplied by the masses of the bodies. So that E and S are attracted to G by a force as (M + E + S) multiplied by their respective distances from G.—Lastly, the times in which these ellipses are described by the bodies and the centres, are all equal by what was before proved respecting motion when the force varies as the distances.

This law of the centripetal force is the only one which preserves the entire ellipticity of the orbits, notwithstanding any mutual disturbances; but it produces, at great distances, motions of enormous velocity. Thus we have seen that Saturn would move at the rate of 75,000 miles in a second (or a third of the velocity of light itself), were there no disturbance from the other bodies; but the disturbance might greatly accelerate this rapid motion. If the law be the inverse square of the distance, there will be a departure from the elliptical form of the orbits and from the proportion of the areas to the times, indicating that the several resulting forces are not directed towards the several centres. But this departure will be less considerable in proportion as the body in the centre of any system, or in the common centre of any number of systems, is of a magnitude exceeding that of the revolving bodies, or systems of bodies, because this will prevent the central body moving far from its place, or much out of a straight line; and also the departure will be less in proportion as the bodies, or systems, revolving are at a great distance from the centres or from the common centre, because the diminution of this distance increases the inclination of the lines in which the disturbing forces act, and thus disturbs the motions of the bodies among themselves. Again, if the law of the attraction varies from the inverse square of the distance in some, and not in others, the disturbing effect will be increased. So that we may infer the universality of the law and also the small amount of the disturbing force, and its acting in nearly parallel lines, if we find the ellipticity of the orbits not much deranged, and the proportions of the areas to the times not greatly interrupted.

Newton proceeds to examine more minutely the disturbances caused in a system of three bodies, of which two smaller ones move round a third larger one, and all attract one another by forces inversely as the squares of the distances. Let S attract M with a force inversely as the square of the distance;



call the mean distance = 1; the mean force will be $\frac{1}{1^2} = 1$. Let the distance from S, successively taken by M in moving round E, or its true distance, be S M, thence the force at M is $\frac{1}{SM^2}$. Take $SL = \frac{1}{SM^2}$, and drawing L N parallel to M E, the forces at M are L N and S N, or L N + M E and S N. Now $LN : ME :: SL : SM$ and $LN = \frac{SL \; ME}{SM}$ $= \frac{ME}{SM^3}$. Therefore the force acting upon M towards E is as $ME + \frac{ME}{SM^3}$, consequently it will increase the attraction of E, but it will not be inversely as the square of the distance; and therefore M will not describe an ellipse round E, and the force N S does not tend towards E, nor does the force resulting from compounding L N, or M E, or L N + M E, with N S, tend to E. So that the areas will not be proportional to the times. Therefore, also, this deviation from the elliptical form and from the proportional description of the areas will be the greater, as the distances L N and N S are smaller. Again, let S attract E with a force as $\frac{1}{S E^2}$; if this were equal to S N, it would, by combining with S N, that is, with the attraction of S on M, produce no alteration in the relative motion of M and E. Therefore, that alteration is only caused by the difference between S N and $\frac{1}{S E^2}$; wherefore the nearer S N is to the proportion of $\frac{1}{S E^2}$, that is (because of the proportion of $S L = \frac{1}{S M^2}$), the nearer S N is to unity, the mean force upon M, and the nearer the forces exerted by S on M and on E approach to equality, the less will the elliptical orbit be disturbed, and the more nearly will the areas be described proportionally to the times. If the disturbing force of S acts in a plane different from that in which M and E are, M will be deflected from the plane of its orbit, because the force $S\,N - \frac{1}{S\,E^2}$ will not pass through E; consequently this deflection will be greater or less in proportion as this difference is greater or less, and will be least when $\frac{1}{S\,E^2}$ is nearly equal to the mean force of S upon M.

We have hitherto been supposing S the greater body round which M and E revolve to be at rest while they revolve round each other (the case of the earth and of other planets having satellites.) If we now suppose E the greater and central body, and that M and S both move round E (the case of the planets round the sun), a similar proposition may be demonstrated with respect to the disturbances; and it is further clear in this case that if S moves round G, the centre of gravity of M and E, the orbit of S will be less drawn from the elliptical form, and its radius vector will describe areas more nearly proportional to the times than if it moved round E. This appears clearly from observing that the direction of the centripetal force towards G, that is S G, must be nearer E than M; that the attractive forces by which S is drawn are as $\frac{1}{S\,M^2} - \frac{1}{S\,E^2}$; that their resulting force lies in the line S G; and also that S M varies, while S E remains the same, or nearly so.

In all these cases the absolute attractive forces are as the masses of the attracting bodies; and if there are a number of these, A, B, C, E, &c., of which A attracts all the rest with forces as $\frac{1}{D^2}$, $\frac{1}{d^2}$, &c., (D, *d*, &c., being the distances from A,) and B also attracts A, C, E, &c., as $\frac{1}{D^2}$, $\frac{1}{d^2}$, &c., the absolute attraction of A and B towards each other are as the masses A and B. Hence in a system, as of a planet and its satellites, if the latter revolve in ellipses, or nearly so, and describe areas proportional, or nearly so, to the times, the forces are mutually as the masses of the bodies; and conversely if the forces are proportional to the masses, and ellipses are described and areas as the times, the mutual attractions of all are inversely as the squares of the distances.

It is proved, by reasoning of the same kind, that the disturbing force of S is greatest when M is in the points C and D of the orbit (or the *quadratures*), and least when M is in A and B (or the line of conjunction and opposition called the syzygies). When M is moving from C to A and from D to B, the disturbing force accelerates the motion of M, which then moves along with the disturbing force. When M moves from A to D, and from B to C, its motion is retarded, because the disturbing force acts against the direction of M's motion. So M moves more swiftly in syzygies than in quadrature, and its orbit is less curved in quadrature than in syzygies; but it will recede further from E in quadrature, unless the eccentricity of the orbit should be such as to counterbalance this recession, for the operation of the combined forces is two-fold; it both makes the line of apsides move forward in one point of the body's revolution and backward in another, but more forward than backward, and so upon the whole makes it advance somewhat each revolution (as we before saw); and it also increases the eccentricity of the orbit between quadrature and syzygy, and diminishes that eccentricity between the syzygy and quadrature. So of the inclination of the orbit, which is always diminished between quadrature and syzygies, and increased between syzygy and quadrature, and is at the minimum when the nodes are in quadrature and the body itself in syzygy.

We found before that the force L N was as $\frac{\text{M E}}{\text{S M}^3}$. The forces L N and N E are directly as the mass S, and when S is very distant, the forces L N and N E vary as $\frac{\text{S}}{\text{S E}^3}$, or inversely as the squares of the periodic times; and if at a given distance the absolute disturbing force be as the magnitude of the disturbing body, whose diameter is d, these forces are as $\frac{d^3}{\text{S E}^3}$; or as the cube of the apparent diameter of S. Also if instead of one satellite, M, moving round E, we have several whose orbits are nearly of the same form or inclination (like the first three of Jupiter), the mean motion of their apsides and nodes each revolution are directly as the squares of their periodic times, and inversely as the squares of the planet's time, and the two motions (apsides and nodes) are to one another in a given ratio.

We now have one of those extraordinary instances which abound in his writings, of Sir Isaac Newton's matchless power of generalization; of apprehending remote analogies, and thereby extending the scope of his discoveries. Having shown how the disturbing forces of bodies in a system act upon their motions with respect to each other, he now examines the effect of such forces upon the constitution of the bodies themselves. He supposes, for example, that a number of masses of a fluid revolve round E at equal distances from it by the same laws of attraction by which M moves round E, and that these masses are thus formed into a ring; then it follows that the portions of this ring will move quicker in syzygy than in quadrature, that is, quicker at A and B than at C and D; also, that the nodes of the ring, or the intersections of its plane with the plane S E, will be at rest in syzygy, and move quickest in quadrature, and that the ring's axis will oscillate as it revolves, and its inclination will vary, returning to its first position, unless so far as the precession of the nodes carries it forward. Suppose now E to be a solid body with a hollow channel on its surface, and that E increased in diameter until it meets the ring, which now fills that channel, and suppose E to revolve round its own axis—the motion of the fluid, alternately accelerated and retarded (as we have shown), will differ from the equable rotatory motion of the solid on its axis, being quicker than the globe's motion in syzygy, and slower in quadrature. If S exerts no force, the fluid will not have any ebbs and flows, but move as round a centre that is at rest; but if the varying attraction of S operates, being greater when the distance is less, the disturbing force acting in the direction S L, and being as $\frac{1}{SM^2}$, will raise the fluid in A and B, or in syzygy, and from thence to quadrature, C and D, while the force L N will depress it in quadrature, C and D, and from thence to syzygy, A and B. If we now suppose the ring to become solid, and the size of E to be again reduced, the inclination of the ring will vary, and oscillate; and the precession of its nodes will continue the same—and so would the globe, if, without any ring at all, it had an accumulation of matter in the equator, or had matter of greater density there than elsewhere, and at the poles. If, on the other hand, there is more matter at the poles, or matter of a less dense kind at the equator, the nodes will advance instead of receding. So that by knowing the motion of the nodes, we can estimate the constitution of the globe; and a perfectly spherical and homogeneous globe will move equally, and with a single motion only round its axis. No other will.

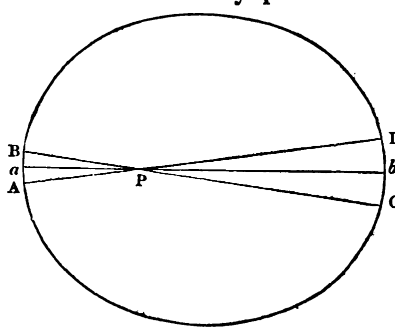
The Sixty-sixth Proposition, or rather its twenty-two corollaries, constitute perhaps the most extraordinary portion of the Principia. We have seen that Sir Isaac Newton here deduces most of the leading disturbances in the motions of three bodies, for example, the moon, earth, and sun, from the propositions which had been before demonstrated. We perceive in succession the motion of longitude and latitude; the various annual equations, motion of the apsides (in which, however, by omitting the consideration of the tangential force, he calculated the amount at one half its true value), the evection,* the alteration, and inclination; the motion of the nodes. Even the doctrine of the tides, and the precession of the equinoxes, are all handled clearly, though concisely, in this proposition. The greater part of the Third Book is occupied with the application of these corollaries to the actual case of the moon, earth, and sun; and it is not any exaggeration to affirm that the great investigations which

* Laplace has erroneously stated that Newton overlooked the Evection; but it forms, though not by name, the subject of the ninth corollary to this Sixty-sixth Proposition.

have been undertaken since the time of Sir Isaac Newton, and of which we have just been surveying the principal results, are an application of the improved calculus to continue the inquiries which he thus here began.

The propositions respecting the masses of the attracting bodies which we considered before the corollaries to the Sixty-sixth Proposition (although they come later in the Principia), and the latter of those corollaries, naturally lead to the subject of the next two sections, the one upon the attraction of spherical bodies, the other upon that of bodies not spherical.

i. The attraction exerted by spherical surfaces and



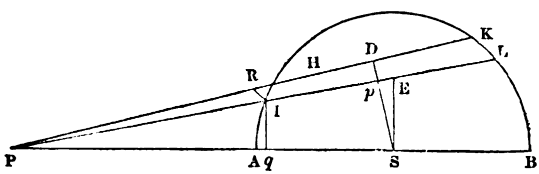
by hollow spheres is first considered. If P be a particle situated anywhere within A B D C, and we conceive two lines A D, B, C, infinitely near each other drawn through P to the surface, and if these lines revolve round $a\,P\,b$, which passes from the middle points a and b, of the small arcs D C and A B, through P, there will two opposite cones be described; and the attraction of the small circles D C, A B upon P, will be in the lines from each point of those circles to P, of which lines C P, D P, are two from one circle, and A P, B P, two from the other circle. Now this attraction of the circle C D is to that of the circle A B, as the circle C D to the circle A B, or as $C\,D^2$ to $A\,B^2$ (the diameters), and by similar triangles $C\,D^2 : A\,B^2 :: P\,C^2 : P\,A^2$. But by hypothesis, the attraction of C D is to that of A B as $A\,P^2 : P\,C^2$; therefore the attraction of D C is to the opposite attraction of A B as $A\,P^2$ to $P\,C^2$, and also as $P\,C^2$ to $A\,P^2$, or as $A\,P^2 \times P\,C^2$ to $A\,P^2 \times P\,C^2$, and therefore those attractions are equal; and being opposite they destroy one another. In like manner, any particle of the spherical surface on one side of P, acting in the direction a P, is equal as well as opposite to the attraction of another particle acting on the opposite side, and so the whole action of every one particle is destroyed by the opposite action of some other particle; and P is not at all attracted by any part

U 2

of the spherical surface; or the sum of all the attractions upon P is equal to nothing. So of a hollow sphere; for every such sphere may be considered as composed of innumerable concentric spherical surfaces, to each of which the foregoing reasoning applies; and consequently to their sum.

We may here stop to observe upon a remarkable inference which may be drawn from this theorem. Suppose that in the centre of any planet, as of the earth, there is a large vacant spherical space, or that the globe is a hollow sphere; if any particle or mass of matter is at any moment of time in any point of this hollow sphere, it must, as far as the globe is concerned, remain for ever at rest there, and suffer no attraction from the globe itself. Then the force of any other heavenly body, as the moon, will attract it, and so will the force of the sun. Suppose these two bodies in opposition, it will be drawn to the side of the sun with a force equal to the difference of their attractions, and this force will vary with the relative position (configuration) of the three bodies; but from the greater attraction of the sun, the particle, or body, will always be on the side of the hollow globe next to the sun.

Now the earth's attraction will exert no influence over the internal body, even when in contact with the internal surface of the hollow sphere; for the theorem which we have just demonstrated is quite general, and applies to particles wherever situated within the sphere. Therefore, although the earth moves round its axis, the body will always continue moving so as to shift its place every instant and retain its position towards the sun. In like manner, if any quantity of movable particles, thrown off, for example, by the rotatory motion of the earth, are in the hollow, they will not be attracted by the earth, but only towards the sun, and will all accumulate towards the side of the hollow sphere next the sun. So of any fluid, whether water or melted matter in the hollow, provided it do not wholly fill up the space, the whole of it will be accumulated towards the sun. Suppose it only enough to fill half the hollow space; it will all be accumulated on one side, and that side the one next the sun; consequently the axis of rotation will be changed and will not pass throngh the centre, or even near it, and will constantly be altering its position. Hence we may be assured that there is no such hollow in the globe filled with melted matter, or any hollow at all, inasmuch as there could no hollow exist without such accumulations, in consequence of particles of the internal spherical surface being constantly thrown off by the rotatory motion of the earth.



If A H K be a spherical section (or great circle), P R K and P I L lines from the particle P, and infinitely near each other, S D, S E perpendiculars from the centre, and I q perpendicular to the diameter; then, by the similar triangles P I R, P p D, we find that the curve surface bounded by I H, and formed by the revolution of I H K H L I round the diameter A B, and which is proportional to I H $\times$ I q, is as $\frac{\mathrm{I\,P}^2}{\mathrm{P\,R} \times \mathrm{P\,S}}$; and if the attraction upon the particle P is as the surface directly, and the square of the distance inversely, or $\frac{1}{\mathrm{P\,I}^2}$, that attraction will be as $\frac{1}{\mathrm{P}p \times \mathrm{P\,S}}$. But if the force acting in the line P I is resolved into one acting in P S and another acting in S D, the force upon P will be as $\frac{P\,q}{P\,I}$, or (because of the similar triangles P I Q, P S p) as $\frac{P\,p}{P\,S}$. The attraction, therefore, of the infinitely small curvilinear surface formed by the revolution of I H is as $\frac{P\,p}{P\,p \times P\,S^2}$ or as $\frac{1}{P\,S^2}$; that is inversely as the square of the distance from the centre of the sphere. And the same may be shown of the surface formed by the revolution of K L, and so of every part of the spherical surface. Therefore the whole attraction of the spherical surface will be in the same inverse duplicate ratio.

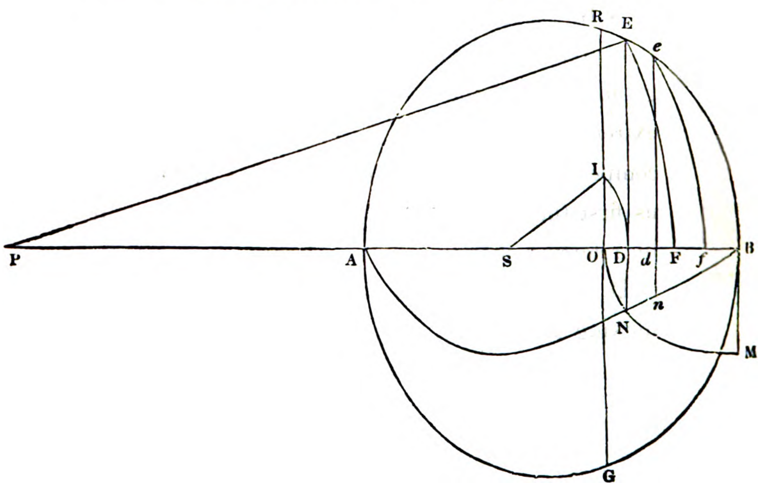
In like manner, because the attraction of a homogeneous sphere is the attraction of all its particles, and the mass of these is as the cube of the sphere's diameter D, if a particle be placed at a distance from it in any given ratio to the diameter, as $m \,.\, D$, and the attraction be inversely as the square of that distance, it will be directly as D^3, and also as $\frac{1}{m^2\,D^2}$, and therefore will be in the simple proportion of D, the diameter. Hence if two similar solids are composed of equally dense matter, and have an attraction inversely as the square of the distance, their attraction on any particle similarly placed with respect to them will be as their diameters. Thus, also, a particle placed within a hollow spheroid, or in a solid, produced by the revolution of an ellipsis, will not be attracted at all by the portion of the solid between it and the surface, but will be attracted towards the centre by a force proportioned to its distance from that centre.

It follows from these propositions, *first,* that any particle placed within a sphere or spheroid, not being affected by the portion of the sphere or spheroid beyond it, and being attracted by the rest of the sphere, or spheroid in the ratio of the diameter, the centripetal force within the solid is directly as the distance from the centre;—*secondly,* that a homogeneous sphere, being an infinite number of hollow spaces taken together, its attraction upon any particle placed without it is directly as the sphere, and inversely as the square of the distance;—*thirdly,* that spheres attract one another with forces proportional to their masses directly, and the squares of the distances from their centres inversely;—*fourthly,* that the attraction is in every case as if the whole mass were placed in the central point;—*fifthly,* that though the spheres be not homogeneous, yet if the density of each varies so that it is the same at equal distances from the centre of each, the spheres will attract one another with forces inversely as the squares of the distances of their centres. The law of attraction, however, of the particles of the spheres being changed from the inverse duplicate ratio of the distances to the simple law of the distances directly, the attractions acting towards the centres will be as the distances, and whether the spheres are homogeneous or vary in density according to any law connecting the force with the distance from the centre, the attraction on a particle without will be the same as if the whole mass were placed in the centre; and the attraction upon a particle within will be the same as if the whole of the body comprised within the spherical surface in which the particle is situated were collected in the centre.

From these theorems it follows, that where bodies move round a sphere and on the outside of its surface, what was formerly demonstrated of eccentric motion in conic sections, the focus being the centre of forces, applies to this case of the attraction being in the whole particles of the sphere; and where the bodies move within the spherical surface, what was demonstrated of concentric motion in those curves, or where the centre of the curve is that of the attracting forces, applies to the case of the sphere's centre being that of attraction. For in the former case the centripetal force decreases as the square of the distance increases; and in the latter case that force increases as the distance increases. Thus it is to be observed, that in the two cases of attraction decreasing inversely as the squares of the central distance (the case of gravitation beyond the surface of bodies), and of attraction increasing directly with the central distance (the case of gravitation within the surface), the same law of attraction prevails with respect to the corpuscular action of the spheres as regulates the mutual action of those spheres and their motions in revolution. But this identity of the law of attraction is confined to these two cases.

Having thus laid down the law of attraction for these more remarkable cases, instead of going through others where the operation of attraction is far more complicated, Sir Isaac Newton gives a general method for determining the attraction whatever be the proportions between the force and the distance. This method is marked by all the geometrical elegance of the author's other solutions; and though it depends upon quadratures, it is not liable to the objections in practice which we before found to lie against a similar method applied to the finding of orbits and forces; for the results are easily enough obtained, and in convenient forms.

If A E B is the sphere whose attraction upon the point P it is required to determine, whatever be the proportion according to which that attraction varies with the distance, and only supposing equal particles of AEB to have equal attractive forces; then from any point E describe the circle E F, and another ef infinitely near, and draw E D, ed ordinates to the diameter A B. The sphere is composed of small concentric hollow spheres $\mathrm{E}\,e\,f\,\mathrm{F}$; and its whole attraction is equal to the sum of their attractions. Now that attraction of $\mathrm{E}\,e\,f\,\mathrm{F}$ is proportional to its surface multiplied by $\mathrm{F}f$, and the angle $\mathrm{DE}r$ being equal to D P E (because $\mathrm{PE}r$ is a right angle by the property of the circle), therefore $\mathrm{E}r = \frac{\mathrm{PE} \times \mathrm{D}d}{\mathrm{DE}}$, and if we call PE, or $\mathrm{PF} = r$, $\mathrm{ED} = y$, and $\mathrm{DF} = x$, $\mathrm{D}d$ will be dx, and $\mathrm{E}r = \frac{r\,dx}{y}$; and the ring generated by the revolution of rE is equal to $r\text{E} \times \text{ED}$, or $r\text{E} \times y$; therefore this ring is equal to $r\,d\,x$, or the attraction proportional to the whole ring Ee will be proportional to the sum of all the rectangles $\text{PD} \times \text{D}d$, or $(a - x)\,d\,x$; that is, to the fluent of this quantity, or to $\frac{2\,a\,x - x^2}{2}$; which by the property of the circle is equal to $\frac{y^2}{2}$. Therefore the attraction of the solid EefF will be as $y^2 \times \text{F}f$, if the force of a particle Ff on P be given; if not, it will be as $y^2 \times \text{F}f \times f$ that force. Now $d\,x : \text{F}f :: r : \text{PS}$, and therefore $\text{F}f = \frac{\text{PS} \times d\,x}{r}$, and the



attraction of $E e f F$ is as $\frac{y^2 \times PS \times dx \times f}{r}$; or taking $f = r^n$ (as any power of the distance P E), then the attraction of $E e f F$ is as $PS . r^{n-1} y^2 \, dx$. Take D N ($= u$) equal to $PS . r^{n-1} y^2$, and let $BD = z$, and the curve B N A will be described, and the fluxional area $N D d n$ will be $n d z =$ (by construction) $PS . r^{n-1} y^2 dx$; consequently $u \, dz$ will be the attractive force of the fluxional solid $E e f F$; and $\int u \, dz$ will be that of the whole body or sphere A E B, therefore the area $ANB = \int u \, dz$ is equal to the whole attraction of the sphere.

Having reduced the solution to the quadrature of A N B, Sir Isaac Newton proceeds to show how that area may be found. He confines himself to geometrical methods; and the solution, although extremely elegant, is not by any means so short and compendious as the algebraical process gives. Let us first then find the equation to the curve A N B by referring it to the rectangular coordinates D N, A D. Calling these y and x respectively, and making $PA = b$, AS (the sphere's radius) $= a$ and P S, or $a + b$, for conciseness, $= \frac{f}{2}$. Then DE^2

$= 2ax - x^2$; $PE = \sqrt{(b+x)^2 + 2ax - x^2} = \sqrt{b^2 + 2\overline{(a+b)}\,x} = \sqrt{b^2 + fx}$; and $DN = y =$ (by construction) $\dfrac{(a+b)\,(2ax - x^2)}{(b^2+fx)^{\frac{n+1}{2}}}$, the attractive force of the particles being supposed as the $\frac{1}{n}$ th power of the distance, or inversely as $(b^2+fx)^{\frac{n}{2}}$. This equation to the curve makes it always of the order $\frac{n+3}{2}$. If then the force is inversely as the distance, A N B is a conic hyperbola; if inversely as the square, it is a curve of the fifth order; and if directly as the distance, it is a logarithmetic; if inversely as the cube, the curve is a conic hyperbola.

The area may next be determined. For this purpose we have $\int y\,dx = \int \dfrac{f(2ax - x^2)\,dx}{2(b^2+fx)^{\frac{n+1}{2}}}$.

Let $2(af + b^2) = h$, this fluent will be found to be $\dfrac{1}{4(a+b)^2} \times \overline{\dfrac{h}{3-n} \times (b^2+fx)^{\frac{3-n}{2}} - \dfrac{b^2}{1-n}}$

$$\times\ (2a+b)^2\,(b^2+fx)^{\frac{1-n}{2}} - \frac{(b^2+fx)^{\frac{5-n}{2}}}{5-n}$$

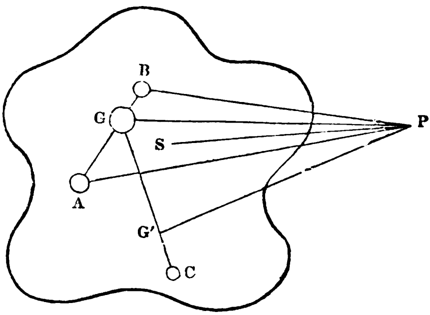
$+$ C; and the constant C is $\frac{1}{4(a+b)^2} \times \left(\frac{b^{5-n}}{5-n}\right.$ $\left. + \frac{(2a+b)^2\,b^{3-n}}{1-n} - \frac{h}{3-n}\,b^{3-n}\right)$. To find the attraction of the whole sphere, when $x = 2a$, we have $\frac{1}{4(a+b)^2} \times \left(\frac{h}{3-n}\,(2a+b)^{3-n} - \frac{b^2}{1-n}\right.$ $\times (2a+b)^{1-n} - \frac{(2a+b)^{5-n}}{5-n} + \frac{b^{5-n}}{5-n} + \frac{b^{3-n}}{1-n}$ $\left.\times\ (2a+b)^2 - \frac{h\,b^{3-n}}{3-n}\right)$ for the whole area A N B, or the whole attraction. This in every case gives an easy and a finite expression, excepting the three cases of $n = 1$, $n = 3$, and $n = 5$, in which cases it is to be found by logarithms, or by hyperbolic areas. If P is at the surface, or A P $= b = 0$, and $n = 2$, then the expression becomes as a, that is, as the distance from the centre directly. We may also perceive from the form of the expression, that if n is any number greater than 3, so that $n - 3 = -m$, the terms b^{3-n} become inverted, and b is in their denominator thus: $\frac{(2a+b)^2}{(1-n)\,b^m}$. Hence, if A P $= b = 0$, or the particle is in contact with the sphere, the expression involves an infinite quantity, and becomes infinite. The construction of Sir Isaac Newton by hyperbolic areas leads to the same result for the case of $n = 3$, being one of those three where the above formula fails. At the origin of the abscissæ we obtain, by that construction, an infinite area; and this law of attraction, where the force decreases in any higher ratio than the square of the distance, is applicable to the contact of all bodies of whatever form, the addition of any other matter to the spherical bodies having manifestly no effect in lessening the attraction.

By similar methods we find the attraction of any portion or segment of a sphere upon a particle placed in the centre, or upon a particle placed in any other part of the axis. Thus in the case of the particle being in the centre S, and the particles of the segment R B G attracting with forces as the $\frac{1}{n}$ power of the distance S O or S I, the curve A N B will by its area express the attraction of the spherical segment, if D N or y be taken $= \frac{\mathrm{IO}^2}{\mathrm{SD}} = \frac{(x-a)^2-c^2}{(x-a)^n}$, S O being put $= c$, and A D $= x$, and A S $= a$, as before; $\int y\, dx$ may be found as before by integrating $\frac{(x-a)^2\,d\,x - c^2\,d\,x}{(x-a)^n}$. The fluent is $\frac{(x-a)^{3-n}}{3-n} - c^2\,\frac{(x-a)^{1-n}}{1-n} + C$; and $C = \frac{2\,c^{3-n}}{n^2 - 4\,n + 3}$; and the whole attraction of the segment upon the particle at the centre S is equal to $\frac{a^{3-n}}{3-n} - \frac{c^2\,a^{1-n}}{1-n} + \frac{2\,c^{3-n}}{n^2 - 4\,n + 3}$. Thus if $n = 2$ the attraction is as $\frac{(a-c)^2}{a}$, or as O B^2 directly, and as S B inversely; and if $c = o$, or the attraction is taken at the centre, it is equal to a; and if the attraction is as the distance, or $n = 1$, then the attractive force of the segment is $\frac{1}{4}\,(a^2 - c^2)^2$.

ii. Our author proceeds now to the attractions of bodies not spherical; an inquiry not perhaps, in its greater generality, of so much interest in the science of Physical Astronomy, where the masses which form the subjects of consideration are either spherical, or very nearly spherical, to which our examination has hitherto been confined. But this concluding part, nevertheless, contains some highly important truths available in astronomical science, because it leads, among other things, to determining the attraction of spheroids, the true figures of the planets.

The attractions of two similar bodies upon two similar particles similarly situated with respect to them, if those attractions are as the same power of the distances $\frac{1}{n}$, are to one another as the masses directly, and the n^{th} power of the distances inversely, or the n^{th} power of the homologous sides of the bodies; and because the masses are as the cubes of these sides, S and s, the attractions are as $S^3 . s^n : s^3 . S^n$, or as $s^{n-3} : S^{n-3}$. Therefore if $n = 1$, the attraction is as $S^2 : s^2$; if the proportion is that of the inverse square of the distance, the attraction is as $S : s$; if that of the cube, the attraction is as $1 : 1$, or equal; if as the biquadrate, the attraction is as $s : S$; and so on: and thus the law of the attractive force may be ascertained from finding the action of bodies upon particles similarly placed.

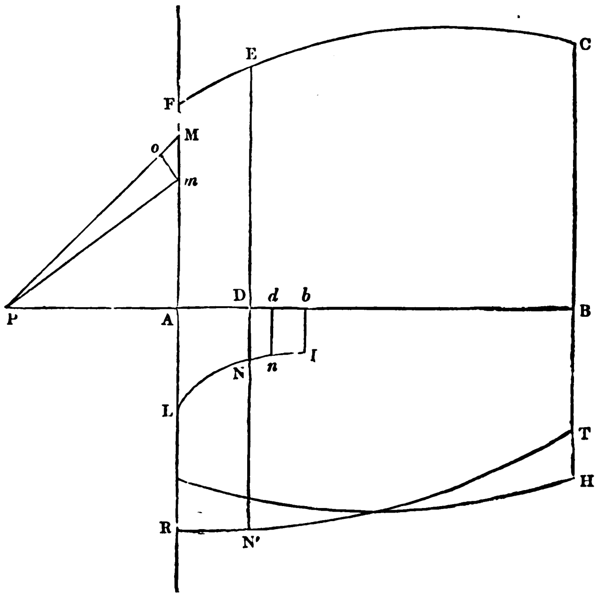
Let us now consider the attraction of any body, of what form soever, attracting with force proportioned to the distance towards a particle situated beyond it. Any two of its particles A B attract P, with forces as $A \times AP$ and $B \times BP$, and if G is their common centre of gravity, their joint attraction is as $(A+B) \times G P$, because B P, being resolved into B G and G P, and A P into A P and G P,



and (by the property of the centre of gravity) $G P \times A = A P \times G$, therefore the forces in the line A P destroy each other, and there remain only $P G \times B$ and $P G \times A$ to draw P, that is $(A + B) \times P G$; and the same may be shown of any other particles C and the centre G′ of gravity, of A, C, and B, the attraction of the three being $(A+B+C) \times G' P$. Therefore the whole body, whatever be its form, attracts P in the line P S, S being the body's centre of gravity, and with a force proportional to the whole mass of the body multiplied by the distance P S. But as the mutual attractions of spherical bodies, the attraction of whose particles is as their distance from one another, are as the distances between the centres of those bodies, the attraction of the whole body A B C is the same with that of a sphere of equal mass whose centre is in S, the body's centre of gravity. In like manner it may be demonstrated that the attraction of several bodies A, B, C, towards any particle P, is directed to their common centre of gravity S, and is equal to that of a sphere placed there, and of a mass equal to the sum of the whole bodies A, B, C; and the attracted body will revolve in an ellipse with a force directed towards its centre as if all the attracting bodies were formed into one globe and placed in that centre.

But if we would find the attraction of bodies whose particles act according to any power n of the distance, we must, to simplify the question, suppose these to be symmetrical, that is, formed by the revolution of some plane upon its axis. Let A M C H G be the solid, M G the diameter of its extreme circle of revolution next to the particle P; draw P M and $p\ m$ to any part of the circle, and infinitely near each other, and take P D = P M, and P o = P m; D d will be equal to o M ($d\ n$ being infinitely near D N), and the ring formed by the revolution of M m

round A B will be as the rectangle A M $\times$ M m, or (because of the triangles A P M, $m\,o$ M, being



similar, and D $d = o$ M) P M $\times$ D d, or P D $\times$ D d. Let D N be taken $= y =$ force with which any particle attracts at the distance P D $=$ P M $= x$, that is as x^n; and if A P $= b$, the force of any particle of the ring is as $\frac{b\,y}{x}$, and the attraction of the ring, described by M m, is as $\frac{b\,y}{x} \times$ D $d \times$ P D, or as $b\ y\ d\ x$, and the whole attraction of the circle whose radius is A M, being the sum of all the rings, will be as $b \int y\ d\ x$, or the area of the curve L N I, which is found by substituting for y its value in x, that is x^n. This fluent or area is therefore $= b \int x^n\ d\ x$ $= \frac{b\ x^{n+1}}{n+1} + \mathrm{C}$; and $\mathrm{C} = \frac{-\ b^{n+2}}{n+2}$. Also, making P $b =$ P E in order to have the whole area of L N I, which measures the attraction of the whole circle whose radius is F A, we have (x being $=$ P $b = c$) $\frac{b\ c^{n+1}}{n+1} - \frac{b^{n+2}}{n+2}$ for that attraction. Then taking D N′ in the same proportion to the circle D E in which D N is to the circle A F, or as equal to the attraction of the circle D E, we have the curve R N T, whose area is equal to the attraction of the solid L H C F.

To find an equation to this curve, then, and from thence to obtain its area, we must know the law by which D E increases, that is, the proportion of D E to A D, in other words, the figure of the section A F E C B, whose revolution generates the solid.

Thus if the given solid be a spheroid, we find that its attraction for P is to that of a sphere whose diameter is equal to the spheroid's shorter axis, as

$\frac{a.A^2 - D.L}{d^2 + A^2 - a^2}$ to $\frac{a^3}{3d^2}$, A and a being the two semi-axes of the ellipsoid, d the distance of the particle attracted, and L a constant conic area which may be found in each case; the force of attraction being supposed inversely as the squares of the distances. But if the particle is within the spheroid, the attraction is as the distance from the centre, according to what we have already seen.

Laplace's general formula for the attraction of a spherical surface, or layer, on a particle situated (as any particle must be) in its axis, is $\frac{2\pi u\, d u}{r}$ $\int f\, df \times \int df\, F$, in which f is the distance of the particle from the point where the ring cuts the sphere, r its distance from the centre of the sphere, or the distance of the ring from that centre, $d u$ consequently the thickness of the ring, π the semicircle whose radius is unity, and F the function of f representing the attracting force. The whole attraction of the sphere, therefore, is the integral taken from $f = r - u$ to $f = r + u$, and the expression becomes $\frac{2\pi . u\, d u}{r} + \int f\, d f \times$

$\times \int d f$ F with $(r + u)$—$(r - u)$, substituted for f, when f results from this integration. Then let $F = \frac{1}{f^2}$, or the attraction be that of gravitation; the expression becomes $\frac{2\pi . u\, d u}{r} \int f\, d f \times \int \frac{d f}{f^2} =$ $\frac{2\pi . u\, d u}{r} \times \frac{f^2}{2} \times -\frac{1}{f} = \frac{2\pi . u\, d u}{r} \times -$ $\frac{(r + u)-(r - u)}{2} = \frac{2\pi u\, d u}{r} \times -u = -2\pi u^2$ $d u \times \frac{1}{r}$; and the co-efficient of $d\, r$, taking the fluxion with r as the variable, is $+\frac{2\pi u^2 d u}{r^2}$; consequently the attraction is inversely as the square of the distance of the particle from the centre of the sphere, and is the same as if the whole sphere were in the centre.*

The First Book of the Principia concludes with some propositions respecting the motion of infinitely small bodies through media, which attract or repel them in their course, that is to say, of the rays of light, which, according to the Newtonian doctrine, are supposed to be bodies of this kind,

* Méc. Cél. liv. ii. ch. 2. The expression is here developed; but it coincides with the analysis in § 11.

hard and elastic, and moving with such rapidity as to pass through the distance of the sun from the earth, or 95 millions of miles, in seven or eight minutes, that is, with a velocity of above 211,000 miles in a second. Sir Isaac Newton shows that, if the medium through which they pass attracts or repels them from the perpendicular uniformly, they describe a parabola, according to Galileo's law of projectiles; but if the attraction or repulsion be not equable, another curve will be described; yet, that in either case the sine of the angle of incidence (or that made with the plane where they enter the medium), is to the sine of the angle of refraction (or that made with the plane they emerge from) in a given ratio; that the velocities before incidence and after emerging are inversely as the sines of incidence and refraction; and that if the velocity after incidence is retarded, and the line of incidence inclined more towards the plane of the refracting medium, the small bodies will be reflected back at an angle equal to that of incidence.

He then remarks on the inflexion and deflexion which light suffers in passing, not through, but by or near bodies, as discovered by Grimaldi,* and as

* Grimaldi termed it *diffraction*.

confirmed by his own experiments. He shows that the rays are bent most probably in curve lines, the nearest rays towards the bending body, the furthest rays away from it; and he infers that, in refraction and reflexion, a similar curvilinear bending takes place somewhat before the actual point of refraction and reflexion. He further mentions the colours formed by flexion, as three coloured fringes or bands, "tres colorum fascias." I, however, long ago showed (Phil. Trans. 1797, Part II.) that this is not the real fact; having found that a much greater number of these fringes are formed by flexion, and that they are, like the prismatic spectrum, images of the luminous body. This experiment has been repeated by Sir David Brewster and others; nor can any doubt be entertained that there are innumerable fringes decreasing in breadth, and in the breadth of the dark intervals between them, until they become evanescent. But as if it were the fate of all this great man's discoveries, that nothing should ever be added to them but by the use of means which he had himself furnished, it was only by applying a form of experiment which Sir Isaac Newton had used in examining the colours of thick and thin plates, that this important fact was ascertained, he not having subjected the phenomenon first observed by Grimaldi to that mode of investigation.*

The Fourteenth Section concludes with an elegant solution of a local problem in Descartes's Geometry, for finding that form of refracting glasses which will make the rays converge to a given focus, a problem, the demonstration of which Descartes had not given. The brilliant discoveries made by Sir Isaac Newton upon the refrangibility and colours of light, not belonging to dynamics, he pursues the subject no further in this place, having reserved the history of those inquiries for his other great work, the Optics,† perhaps the only monument of human genius that merits a place by the side of the Principia.

* The Undulatory Theory of light, towards which philosophers have of late years appeared to lean, is no exception to this remark; for the principles of that Theory may be found in the Eighth Section of the Second Book of the Principia, and the Scholium which concludes that Section seems to anticipate the application of its principles to Optical Science.

† An abstract of these discoveries had been given in the Lectiones Opticæ at Cambridge seventeen years before the publication of the Principia in 1687. The Optics only appeared in 1704.

The truths which we have been contemplating respecting the attractions of bodies are fruitful in important consequences respecting the constitution of the universe. We have seen that the law of attraction which makes it decrease as the squares of the distances increase, and the law which makes it increase as the distances increase, are the only laws which preserve the proportions between the force and the distance, the same for the attraction of the particles of bodies, and for the attraction of the masses in which those particles may be distributed—the only laws which make the attraction of bodies the same with that of their mass placed in the centre of gravity. Now these two laws regulate the actions of bodies gravitating towards each other, the one being the law of gravitation beyond the surface of attracting bodies, the other, the law of gravitation between the surface and the centre. Thus, then, there is every reason to believe that this law pervades the material world universally, acting in precisely the same manner at the smallest and at the greatest distances, alike regulating the action of the smallest particles of matter, and the mightiest masses in which it exists. This action, too, is everywhere mutual; it is always in direct proportion to the masses of the attracted and attracting bodies at equal distances; where the masses are equal, it is inversely as the squares of the distances beyond the bodies, and within the bodies, as the distances from the centre; and where the masses and distances vary, it is as the masses divided by the squares of the distances in the one case, and as the masses multiplied by the distances in the other. This law then pervades and governs the whole system.

The discoveries which astronomers have made since the death of Newton, upon the more remote parts of the universe, by the help of improvements in optical instruments, have further illustrated the general prevalence of the law of gravitation. The double fixed stars, many of which had long been known to astronomers, and which were believed to retain at all times their relative positions, have now been found to vary in their distances from each other, and to move with a velocity sometimes accelerated, sometimes retarded, but apparently round one another, or rather round their common centres of gravity. A course of observations continued for above twenty years led Herschel to this important conclusion about the year 1803; his son has greatly added to our knowledge of these motions; and Professor Struve, of Dorpat, applying geometrical reasoning to the subject, calculated the orbits in which some of the bodies appear to move. One of the most remarkable is the star γ Virginis, on which Cassini had made observations in 1720.* It has now been found that one of the stars of which it is composed is smaller than the other; that the revolving motions of the two during the first 25 years had a mean annual velocity of 31′ 23″; during the next 21 years, of 29′ 17″; during the next 17 years, of only 2′ 42″; and during the last two years (1822, 23) of no less than 52′ 51″. The elder Herschel calculated the time of their whole revolution, the periodic times of those distant suns, at 708 years; it is now supposed not to exceed 629. Another pair of stars are found to revolve round one another in between 43 and 44 years, while a third pair take 12 centuries to accomplish their revolution.† Although our observations are far too scanty to lay as yet the ground of a systematic theory of these motions, they appear to

* Mém. Acad. des Sciences, 1720.

† Phil. Trans. 1803, p. 339; ib. 1824, Part III.

warrant us in assuming that the law of attraction which governs our solar system extends to those remote regions, and as their suns revolve round one another, each probably carrying about with it planets that form separate systems, we shall probably one day find that equal areas are there as here described in equal times, and that the orbits are elliptical; or, which would come to the same thing, that the sesquiplicate proportion of the periodic times and mean distances is observed;* from whence the conclusion would of necessity follow, that the centripetal force followed the rule of the inverse square of the distance, and that gravitation such as we know it in our part of the universe, likewise prevails in these barely visible regions. Thus additional confirmation accrues to the first great deduction drawn from the theorems respecting attraction in the Principia.

But other interesting corollaries are also to be deduced from these propositions. They enable us to ascertain, for example, the attractions, the masses,

* It may even seem that already the observed axes of those remote orbits, when compared with their periodic times, approach the sesquiplicate ratio. Thus one has its axis 7″·9, and time 58 years; and another its axis 30″·8, and time 452 years.

and the figures of the heavenly bodies. Sir Isaac Newton boldly and happily applied them to determine these important particulars, apparently so far removed beyond the reach of the human faculties.

1. The weights of bodies at the surface of the different planets were thus easily determined. The law by which the attractive force of spherical bodies decreases as the square of the distance increases, whether those bodies be homogeneous or not, provided their densities vary in the same proportion, and the other law regulating the proportion between the periodic times and the distances of the planets, enabled him to compute the attraction of each planet, for equal bodies at given distances from their centres, by comparing the observed distances and periodic times of each; and he was thus also enabled, by knowing their diameters, to ascertain the weights of bodies at their surfaces. He found in this manner, that the same body which at the surface of the Earth weighs 435 pounds, at that of the Sun weighs 10,000, at that of Jupiter 943, and at that of Saturn 549.

2. So too the masses of matter in each planet and in the satellites may be ascertained. The

motions of the satellites of Jupiter and Saturn afford the easiest means of determining the masses of those planets; and the motions of the other planets round the Sun enable us to solve the problem, though not so accurately, as to them. The mass of Jupiter compared with that of the Earth may be easily supposed to be prodigious, when we find all his satellites revolve round him so much more rapidly than the Moon does round the Earth, although all of them but one have much larger orbits. Thus the second satellite revolves in a seventh part of our lunar month, though its path is half as long again: and hence, its velocity is between 10 and 11 times as great. Sir Isaac Newton ascertained the masses of Jupiter, Saturn, and the Earth to be to that of the Sun as $\frac{1}{1067}$, $\frac{1}{3021}$, $\frac{1}{169282}$, to 1 respectively. In like manner the densities are found, being as the weights (first found) divided by the axes. Thus he determined the relative densities of Jupiter, Saturn, and the Earth to be as 94½, 67, and 400, to 100, the density of the Sun. Laplace has ascertained the masses of the heavenly bodies by an entirely different calculus, founded upon the comparison of numerous observations with the formulæ for determining the disturbances. The result is extremely remarkable in one particular. It agrees to a fraction, as regards Jupiter, with the calculation of Newton, making the mass of the planet $\frac{1}{1067}$. But the observations of Pound respecting Saturn's axis, on which Newton had estimated Saturn's mass, were subject to considerable uncertainty; so at least Laplace explains the difference of his own results; but he admits* that even in his day there prevailed considerable uncertainty respecting this planet's mass, while that of Jupiter, being well ascertained, agrees perfectly with Sir Isaac Newton's deduction. Laplace gives the masses of the four great planets thus, that of the sun being unity: Venus $\frac{1}{356632}$; Mars $\frac{1}{2546320}$; Jupiter $\frac{1}{1067 \cdot 09}$; differing by $\frac{1}{11}$ only from Newton's, who indeed did not insert decimals at all); and Saturn $\frac{1}{3534 \cdot 08}$.† The Moon's mass he makes $\frac{1}{68 \cdot 5}$, that of the Earth being unity, while the greatest of Jupiter's satellites is only 0,0000884972, Jupiter being unity. This

* Méc. Cél. liv. vii. ch. 16, s. 44.

† Méc. Cél. liv. x. ch. 8, 9; correcting liv. vi. ch. 6.

great geometrician's observations upon Saturn's ring are peculiarly worthy of attention. The extreme lightness of the matter of which the planet consists, has already been shown; it is six times lighter than the mean density of the Earth; or, if the mean specific gravity of the latter be taken as 5,* that of water being as 1, the matter of which Saturn is composed must be only 3½ times heavier than cork, and lighter than India rubber. But Laplace has satisfactorily shown that his rings must be composed of a fluid, and that no other construction can account for their permanence.†

3. Sir Isaac Newton, lastly, by the principles which we have been explaining in the latter part of our Analysis, investigated the figures of the heavenly bodies. Thus he especially examined that of the Earth. This planet, in revolving round its axis, gives those particles the greatest tendency to fly off which move with the greatest velocity, that is, those which are furthest from their centres of rotation; in other words, those which are nearest the equator; while those near the poles, describing much smaller circles, move much slower and have far less tendency to fly off. Hence there is an accumula-

* The mean of Maskelyne and Cavendish's experiments.

† Méc. Cél. liv. iii. ch. 6.

tion of matter towards the equator, which is raised, while the poles are depressed and flattened, and the equatorial axis is longer than the polar. By comparing the space through which heavy bodies fall in a second in our latitudes with the centrifugal force at the equator, he found that the gravity of bodies there is diminished $\frac{1}{289}$ at least, or that the equatorial axis is, at least, $\frac{1}{289}$ longer than the polar. But he considered this estimate as below the truth, because it does not make allowance for the effect produced on gravitation by the increase of the distance at the equator from the centre. Accordingly, by a skilful application of the method of false position, he corrected this calculation, and ultimately brought out the proportion to be that of 229 to 230, making the equatorial axis about $34\frac{1}{5}$ miles longer than the polar, the whole axis being about 7870 miles. He also estimated the two axes of Jupiter to be as $11\frac{1}{6}$ to $10\frac{1}{6}$, supposing the density of the body to be the same throughout; but if it is greater towards the equator, our author observed that the difference between the axes might be decreased as low as 13 to 12, or even 14 to 13; which agreed well enough with Cassini's observations in those days, and still more nearly with Pound's. But more accurate observation has since shown that the difference is considerably less, the disproportion being not more than that of 1074 to 1000; so that the planet must be very far from homogeneous and its equatorial density greatly exceed its polar. Thus, too, accurate measurements of a degree of latitude in the equatorial and polar regions, and experiments on the force of gravity, as tested by the length of the pendulum vibrating seconds in those different parts of the globe, have led to a similar inference respecting the earth, its axis being now ascertained to bear the relation, not of 230 to 229, as Newton at last concluded, nor even that of 289 to 288, according to his first approximation, but only that of 336 to 335,* being an excess of little more than $23\frac{1}{2}$ miles. The calculation of Newton was formed on the supposition of the Earth being homogeneous; and it is worthy of remark, that although the later observations, by proving the flattening at the poles to be less than he, on this hypothesis, assigned it, have shown the Earth not to be homogeneous, no correction or improvement whatever has been made on his theory in this respect. We find Laplace, on the contrary, in the very passage

* Méc. Cél. liv. iii. ch. 5.

to which we are now referring, assuming his precise fraction $\frac{1}{230}$ as the one given by the theory upon the supposition of the globe being homogeneous, and reasoning upon that fraction.*

Now it is fit that we here pause to contemplate perhaps the most wonderful thing in the whole of the Newtonian discoveries. The subject of curvilinear motion, or motion produced by centripetal forces, was certainly in a great measure new, and Sir Isaac Newton's treatment of it was in the highest degree original and successful. But the laws of attraction, the principles which govern the mutual actions of the planets, and generally of the masses of matter, on each other, was still more eminently a field not merely unexplored, but the very existence of which was unknown. Not only did he first discover this field, not only did he invent the calculus by means of which alone it could be explored, and without which hardly a step could be made across any portion of it (for the utmost resources of geometrical skill in the hands of the Simsons and the Stewarts themselves, who in other inquiries had performed such wonders by ancient analysis, would

* Méc. Cél. liv. iii. ch. 5, s. 41.

have failed to do anything here), but the great discoverer actually completed the most difficult investigation of this new region, and reached to its most inaccessible heights, with a clearness so absolute, and a certainty so unerring, that all the subsequent researches of his followers, and all their vast improvements on his calculus, have not enabled them to correct by the fraction of a cipher his first results. The Ninetieth and Ninety-first Propositions of the First Book, containing the most refined principles of his method, are applied by him in the Nineteenth of the Third Book to the problem of the Earth's figure; his determination of the ellipticity, supposing the mass homogeneous, is obtained from that application. A century of study, of improvement, of discovery has passed away; and we find Laplace, master of all the new resources of the calculus, and occupying the heights to which the labours of Euler, Clairaut, D'Alembert, and Lagrange have enabled us to ascend, adopting the Newtonian fraction of $\frac{1}{230}$, as the accurate solution of this speculative problem. New admeasurements have been undertaken upon a vast scale, patronized by the munificence of rival governments;

new experiments have been performed with improved apparatus of exquisite delicacy; new observations have been accumulated, with glasses far exceeding any powers possessed by the resources of optics in the days of him to whom the science of optics, as well as dynamics, owes its origin; the theory and the fact have thus been compared and reconciled together in more perfect harmony; but that theory has remained unimproved, and the great principle of gravitation, with its most sublime results, now stands in the attitude, and of the dimensions, and with the symmetry, which both the law and its application received at once from the mighty hand of its immortal author.

II.

Hitherto we have considered all motion as performed *in vacuo,* or in a medium which offers no resistance to the action of forces upon bodies moving in any direction. It was necessary that the subject should first be discussed upon this supposition; and the hypothesis agrees with the fact as far as the motions of the heavenly bodies are concerned. But all the motion of which we have any experience upon or near the surface of the earth, is performed in the atmosphere that surrounds our globe; and therefore, as regards all such motion, a material allowance must be made for the resistance of the air when we apply to practice our deductions from the theory. It is also obvious that a still greater effect will be produced upon moving bodies, if their motion is performed in a denser fluid, as water. Further, the pressure and motion of fluids themselves form important subjects of consideration, independent of any motion of bodies through them and impeded by them. These several matters form the subject of the sciences of Hydrostatics, Hydraulics, and Pneumatics; the first treating of the weight and pressure of watery fluids, the second of their motion, the third of aeriform or elastic fluids. They are discussed in the Second book of the Principia. It consists of Nine Sections; of which the First Three discuss the motion of bodies to which there is a resistance in different proportions to the velocity of the motion; the Fourth treats of circular or rather spiral motion in resisting media; the Sixth, of the motion and resistance of pendulums; and part of the Seventh discusses the motion of projectiles; while the rest of the Seventh, and the whole of the four remaining sections, treat of the pressure and motion of fluids themselves and propagated in pulses, or otherwise, through fluids. We shall arrange the subjects under these Five heads, instead of following the precise order of the work itself.*

Two observations are applicable to this branch of the subject, and to the treatment of it in the Principia; and these observations lead to our distinguishing this portion of that great work from the rest.

First. Much more had been accomplished of dis-

* For the arrangement, see the Summary of Contents, vol. i. p. xxxiv, v.

covery respecting the dynamics of fluids before the time of Sir Isaac Newton, in proportion to the whole body of the science, than in the other branches of Mechanics. The Newtonian discoveries, therefore, effected a less considerable change upon this department of Physics than upon Physical Astronomy and the general laws of motion. As early as the time of Archimedes the fundamental principle of the general or undequâque pressure of fluids had been ascertained; many of the easier problems, and even some of the more complicated, had been investigated by its aid. When dynamical science was newly constructed by the illustrious Galileo, the progress which he made may almost be said to have formed Hydrostatics and Hydraulics into a system; and Pascal's original and inventive genius, soon afterwards applied to it, enabled him clearly to perceive the hydrostatic paradox, and even led him to a plain anticipation of the hydrostatic press.* Torricelli about the same period reduced the atmosphere under the power of weight and measure, making it the subject of calculation by the beautiful experiment which first ascertained its gravity, which

* He calls a box of water "a new mechanical principle by which we may multiply force *ad libitum*." (Equil. of Fluids, 1653).

had long been suspected but not proved. Pascal first extended the Torricellian experiment to all the perfection, indeed, which it has ever attained, by showing the connexion between the height of places on the earth's surface, and that of the mercurial column; thus demonstrating satisfactorily the pressure of the atmospherical column. Torricelli had also, from experiments on the spouting of water, inferred that the velocity of the spouting column, or jet, is as the square root of the height of the reservoir of fluid whose pressure causes the flow. So that the fundamental principles being ascertained, considerable progress was also made in their systematic application, when Sir Isaac Newton came to treat the subject as a branch of his general dynamical theory, and to investigate the laws of fluids by means of those profound principles which he had established with respect to all motion. Thus more was done before his time, and less consequently left for him to do here, than in the other branches of the general subject.

Secondly. It is also true that the work which he produced upon this branch of science, did not attain the same perfection under his hands, as the rest of the Principia. Although he treated it upon mathematical principles, he left considerably more to be done by his successors than he left to be added by those who should follow him in the field of Physical Astronomy. A great step was almost immediately made by J. Bernouilli, in ascertaining the effects of the air's resistance upon the motion of projectiles; and an error so apparent was pointed out in one of the Propositions in the Principia (Book II. Prop. 37*), that the correction coming to the author's knowledge, he struck it out of the second edition, then in the press. His original solution of the problem as to spouting columns, having differed from the rule which Torricelli had deduced experimentally, Newton again investigated the question by a different and an admirable process; but even now the subject remains in a very unsatisfactory state. Nor can it be said that the science of hydrodynamics generally has attained the perfection of the other branches of Mechanical philosophy; while it is certain that the application to it of the calculus by Euler, and still more by Clairaut, has greatly added to the theorems left by Sir Isaac Newton; and the researches of Laplace upon capillary attraction, form

* First Edition, published in 1687.

a department of science almost unknown before the latter part of the eighteenth century.

The statement of these particulars was necessary in order to place the relative merits of the different branches of the Principia in their true light. That a great improvement was accomplished in natural knowledge by this portion of Sir Isaac Newton's discoveries, none can doubt. That the Second Book displays at every step the profound sagacity and matchless skill of its author, is undeniable. That it would have conferred lasting renown upon any one but himself, had it been the only work of another man, is certain. Nor can we forget that in rating its importance as we have ventured to do, we only undervalue this portion of the Principia, by applying to it the severest of standards, comparing it with the discovery of the laws which govern the system of the universe, and placing it in contrast with the other parts of that unrivalled effort of human genius.

NOTE.

The argument in page 436 *et seq.*, vol. ii., is succinctly and popularly stated respecting the supposition of a hollow in the centre of the Earth, and several steps are omitted. One of these may be here mentioned in case it should appear to have been overlooked. Suppose a mass m detached from the hollow sphere M, and impelled at the same time with that sphere by an initial projectile force, then its tendency would be to describe an elliptic orbit round the sun, the centre of forces, and if it were detached from the earth it would describe an ellipse, and be a small planet. But as the accelerating force acting upon it would be different from that acting on the earth, the one being as $\frac{S + M}{D^2}$, and the other as $\frac{S + m}{D^2}$ (D being the distance and S the mass of the sun), it is manifest that, sooner or later, its motion being slower than that of the hollow sphere, if m be placed in the inside, it must come in contact with the interior circumference of the space, and either librate, or, if fluid, coincide with it, as assumed in the text. Where parts of the spherical shell come off by the centrifugal force, of course no such step in the reasoning is wanted; nor is it necessary to add that neither those parts nor any other within the hollow shell can have any rotatory motion.

THE END.

ERRATA.

Page	Line	
31,	15,	*after* "nor" *insert* "is."
ib.	18,	*for* "no" *read* "not."
74,	1,	*for* "squares" *read* "square."
107,	7,	*after* "being" *insert* "without."
214,	3,	from bottom, *for* "Junia" *read* "Simia."
347,	14,	*for* "C, P," *read* "C. F."
376,	13,	*for* "veloci" *read* "velocior."
410,	10,	from bottom, *for* "the" *read* "their."
ib.	9,	from bottom, *for* "are commensurable" *read* "should be commensurable, which those of Jupiter and Saturn are, after" &c.
411,	7,	from bottom, *for* "expression by which" *read* "equation to."
ib.		last line, *for* "8S5" *read* § 65.
415,	6,	from bottom, *for* "109,000" *read* "109,060."
416,		last line, dele &c. in the denominator of $\frac{d\,e'\,\text{sin.}\,\varpi'}{d\,t,\ \&c.}$.
421,	10,	from bottom, *after* "evanescent" *insert* "disturbing."
425,		letter S should be put in the centre of the circle.
ib.	6,	*for* "L N + M E" *read* "L N + Q, Q being a quantity that varies as $\frac{1}{M\,E^2}$."
429,	8,	*for* "less curved" *read* "more curved."
ib.	1, 7, 9,	*for* "syzygies" *read* "syzygy."

London: Printed by W. Clowes and Sons, Stamford-street.